QUANTUM FIELDS, STRINGS, & BLACK HOLES
a primer for nonexperts

(Based on lectures delivered at the ICTP)

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Preface

Study of black holes in string theory has revealed a beautiful and precise connection between the physics of quantum black holes and topics in number theory and geometry. The aim of these lectures is to outline these connections for an audience of mathematicians through illustrative examples starting with basic concepts and motivations from physics.

Our goal is to give a sufficiently detailed introduction to some of the important concepts such as quantum ensembles, entropy, quantum fields, black holes, event horizon, supersymmetry, conformal field theory. Where possible, we use simple illustrative examples with explicit computations which capture the essential concepts.

Quantum Black Holes

A black hole is at once the most simple and the most complex object.

It is the most simple in that it is completely specified by its mass, spin, and charge. This remarkable fact is a consequence of a the so called 'No Hair Theorem'. For an astrophysical object like the earth, the gravitational field around it depends not only on its mass but also on how the mass is distributed and on the details of the oblate-ness of the earth and on the shapes of the valleys and mountains. Not so for a black hole. Once a star collapses to form a black hole, the gravitational field around it forgets all details about the star that disappears behind the even horizon except for its mass, spin, and charge. In this respect, a black hole is very much like a structure-less elementary particle such as an electron.

And yet it is the most complex in that it possesses a huge entropy. In fact the entropy of a solar mass black hole is enormously bigger than the thermal entropy of the star that might have collapsed to form it. Entropy gives an account of the number of microscopic states of a system. Hence, the entropy of a black hole signifies an incredibly complex microstructure. In this respect, a black hole is very unlike an elementary particle.

Understanding the simplicity of a black hole falls in the realm of classical gravity. By the early seventies, full fifty years after Schwarzschild, a reasonably complete understanding of gravitational collapse and of the properties of an event horizon was achieved within classical general relativity. The final formulation began with the singularity theorems of Penrose, area theorems of Hawking and culminated in the laws of black hole mechanics.

Understanding the complex microstructure of a black hole implied by its entropy falls in the realm of quantum gravity and is the topic of present lectures. Recent developments have made it clear that a black hole is 'simple' not because it is like an elementary particle, but rather because it is like a statistical ensemble. An ensemble is also specified by a few conserved quantum numbers such as energy, spin, and charge. The simplicity of a black hole is no different than the simplicity that characterizes a thermal ensemble.

Quantum properties of black holes are of great significance for contemporary research in quantum gravity[1]. One of the outstanding problems in twentieth century physics is to develop a consistent framework of Quantum Gravity that unifies General Relativity with Quantum Mechanics. In any purported theory of quantum gravity, it is essential that there is a way to understand the quantum properties of black holes in statistical terms consistent with Boltzman relation.

Superstring theory is the most promising candidate for such a unification. In superstring theory there has been important progress in understanding the entropy of a class of black holes in terms of its microstates consistent with the Boltzmann relation. In the absence of direct experimental probes to explore the theory at the required energies, the quantum structure of black holes provides a very valuable window into the short-distance structure of quatum gravity.

[1] *Classical* black holes are also of enormous importance in astrophysics, especially after the recent detection of gravitational waves emanating from a pair of black holes zzz.

We summarize below how the black hole connects quantum mechanics with general relativity.

Quantum Gravity

Quantum Mechanics	$\leftrightarrow$	General Relativity
Statistical Mechanics	$\leftrightarrow$	Thermodynamics
Quantum Ensemble	$\leftrightarrow$	Entropy
Enumerative Geometry	$\leftrightarrow$	Black Hole
Hilbert Space	$\leftrightarrow$	Spacetime Geometry
Modular Forms	$\leftrightarrow$	Hardy-Ramanujan-Rademacher
Conformal Field Theory	$\leftrightarrow$	Anti de Sitter Spacetime

Table 1: Quantum Gravity

A Physics-Maths Dictionary

We give below a 'dictionary' for some of the key physical concepts.

Physical system	$\leftrightarrow$	Hilbert space $\mathcal{H}$
State	$\leftrightarrow$	Vector $\vert\psi\rangle$
Observable	$\leftrightarrow$	Self-adjoint operator A
Conserved observable	$\leftrightarrow$	$[H, A] = 0$
Quantum number	$\leftrightarrow$	Eigenvalue α
Symmetry	$\leftrightarrow$	Group
Lorentz group	$\leftrightarrow$	$SO(1, d)$
Poincaré group	$\leftrightarrow$	$ISO(1, d) = \mathbb{R}^{1,d} \rtimes SO(1, d)$
Supersymmetry	$\leftrightarrow$	Supergroup
Supersymmetry algebra	$\leftrightarrow$	$\mathbb{Z}_2$-graded Lie algebra

Table 2: Dictionary

Chapter 1

Quantum Mechanics

Perhaps the most important physical principle is the *Atomic Hypothesis* which states that *all physical processes can be understood in terms of motion of 'atoms'.* The notion of an 'atom' as the indivisible unit of matter has evolved over the centuries. In the past, an atom signified a particle like a dust particle; in early twentieth century it signified an atom like the hydrogen atom; now it signifies an *elementary particle* like an electron or a photon. Quantum mechanics is the framework for describing the dynamical motion of atoms interacting with each other.

From a modern perspective, an elementary particle is best understood as the indivisible 'quantum' of energy of a *quantum field*. Thus, a photon is the quantum of the electromagnetic field and the electron is the quantum of a Dirac electron field. From this perspective, the atomic hypothesis can now be recast to state that *all physical processes can be understood in terms of dynamics of quantum fields.*

The theory of quantum fields has proved to be an immensely successful framework for describing all known *non-gravitational* physical processes over a very broad range of distance scales including all of chemistry, all of atomic and nuclear physics, all the way to very short distances probed in the Large Hadron Collider. It is hardly possible to explain quantum field theory in a few lectures. See for a course taught by physicists aimed at mathematicians.

From a pedagogical point of view, it is possible to explain a large part of quantum field theory by studying a much simpler quantum mechanical system–the quantum harmonic oscillator. One can go surprisingly far by thinking of a quantum field as a collection of quantum oscillators and understand most of the essential ideas underlying quantum field theory. As we explain in §2, various Fourier modes of a quantum field perform 'harmonic motion'; thus a quantum field be viewed as a collection of quantum harmonic oscillators interacting with each other. For this reason, a quantum oscillator is a system of fundamental importance in physics. This is the point of view that we will develop in these lectures. It turns out that quantum oscillators are also the simplest way to illustrate many of the interesting connections between physics and topics in combinatorics, enumerative geometry, and number theory.

1.1 Axioms of Quantum Mechanics

Quantum mechanics is a framework for describing the *dynamics* or equivalently the *time evolution* of an isolated physical system whose parts may be dynamically interacting with each other. An isolated physical system can be a single elementary particle or a collection of atoms in a star or the air in this room.

1. The *state* of every isolated 'physical system' S is represented by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ with unit norm $\langle\psi|\psi\rangle = 1$[1]. Two states related by a phase multiplication are physically equivalent, hence $|\psi\rangle \sim e^{i\theta}|\psi\rangle$.

2. Every *observable* of the physical system is represented by a self-adjoint operator on $\mathcal{H}$.

3. There is a preferred observable H for every system called the *Hamiltonian* of the system which measures the total *energy* of the system.

4. The *time evolution* of a physical observable $\mathcal{O}(t)$ is determined by the Heisenberg equation

   $$i\frac{dA(t)}{dt} = -[H, A(t)]\,, \tag{1.1}$$

   where the *commutator* of two operators A and B is defined as usual by

   $$[A, B] := AB - BA\,. \tag{1.2}$$

   This equation can be readily solved by defining the time-evolution operator $U(t)$:

   $$U(t) := e^{-iHt}\,, \qquad i\frac{dU(t)}{dt} = [H, U(t)]\,, \qquad U(0) = \mathbf{1}\,. \tag{1.3}$$

[1] In Dirac's notation commonly used in physics literature, the inner product of two vectors $|\phi\rangle$ and $|\psi\rangle$ is denoted by $\langle\phi|\psi\rangle$. We explain this notation in the next subsection.

Then the time-evolution of the observerable is implemented by the adjoint action of a unitary operator:

$$A(t) = U^{\dagger}(t)A(0)U(t)\,. \tag{1.4}$$

This implies that the time evolution in quantum mechanics is *unitary.* It is evident from (1.1) that the Hamiltonian is independent of time and is 'conserved' because it commutes with itself. This corresponds to the conservation of energy in the physical system under consideration[2].

5. If the system is in a state $|\psi\rangle$ which is an eigenvector of an observable A with eigenvalue α, then a physical *measurement* of the observable yields a number equal to that eigenvalue with unit probability . Since the observable is self-adjoint, $A^{\dagger} = A$, all its eigenvalues $\{\alpha_i\}$ are real.

Comment 1: Consider two non-interacting systems S_1 and S_2 with their respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and Hamiltonians H_1 and H_2. For the combined system of $S = S_1 \bigcup S_2$, the Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The total Hamiltonian is $H = \mathbf{1} \otimes H_1 + H_2 \otimes \mathbf{1}$ which is often written as $H = H_1 + H_2$. If the two systems are 'interacting' then the total Hilbert space can no longer be thought of as a product space.

Comment 2: A state $|\psi\rangle$ may not be an eigenstate of the observable A. In this case, the result of a measurement can yield any of the eigenvalues of A. If P_i is the projection operator onto the eigen-subspace with eigenvalues α_i, then the probability of obtaining α_i as the result of a measurement is given by the expectation value $\langle\psi|P_i|\psi\rangle$. After the measurement, the state $|\psi\rangle$ 'collapses' onto the state $P_i|\psi\rangle$. The measurement axiom is one of the most subtle and much debated axioms especially because of the need for a collapse of the state after a measurement. In a more satisfactory formulation, it should be possible to describe the measurement process entirely in terms of a unitary evolution of the combined system including the measuring apparatus.

Comment 3: Consider a set of mutually commuting self-adjoint operators $\{A^{(1)}, A^{(2)}, \ldots, A^{(n)}\}$. Since they are commuting, they can be diagonalized simultaneously. Such a set of operators is called a *complete* set of commuting observables if every state $|\psi\rangle \in \mathcal{H}$ can be uniquely labelled by the eigenvalues of these operators as $|\psi\rangle = |\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(3)}\rangle$.

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[2] The energy E in a given eigenstate $|E\rangle$ of the Hamilotonian is the eigenvalue of the Hamiltonian. The Hamiltonian is assumed to be a bounded operator so that $E \geq E_0$.

1.2 A Two-state System

To illustrate the essential ideas of quantum mechanics, we consider a particularly simple physical system whose Hilbert space is two-dimensional and whose Hamiltonian in the diagonal basis is given by

$$H = E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.5}$$

which by itself furnishes a complete set of commuting observables. Hence, we label the orthonormal basis vectors by (the sign of) their H eigenvalues

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{1.6}$$

A general state vector with unit norm can thus be represented by

$$|\psi\rangle = a|+\rangle + b|-\rangle \qquad \textit{with} \qquad |a|^2 + |b|^2 = 1\,. \tag{1.7}$$

If we measure the energy of the system in identically prepared copies all in the state $|\psi\rangle$), then we will obtain the result E with probability $|a|^2$ and $-E$ with probability $|b|^2$. Since $|\psi\rangle$ has unit norm, probabilities add up to one.

In Dirac's notation, the column vector $|\psi\rangle$ is referred to as the *ket* vector

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.8}$$

the Hermitian conjugate row vector is referred to as the *bra* vector.

$$\langle\psi| = \begin{pmatrix} a^* & b^* \end{pmatrix} \tag{1.9}$$

The inner product of two vectors

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} , \qquad |\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix} \tag{1.10}$$

in $\mathcal{H}$ is then denoted by the *bracket*:

$$\langle\phi|\psi\rangle = \begin{pmatrix} c^* & d^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = c^*a + d^*b. \tag{1.11}$$

One can denote the usual matrix multiplication of a column vector and a complex conjugated row vector by

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c^* & d^* \end{pmatrix} = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix} . \tag{1.12}$$

An orthonormal basis of vectors $\{|i\rangle\}$ satisfies the orthonomality

$$\langle i|j\rangle = \delta_{ij} \tag{1.13}$$

and the completeness relation

$$\sum_i |i\rangle\langle i| = \mathbf{1}\,. \tag{1.14}$$

Any observable A of the two-state system can be expressed as a linear combination with real coefficients of the identity $\mathbf{1}$ and the three self-adjoint Pauli spin matrices $\{\sigma_i\}$, $(i = 1, 2, 3)$ with the matrix reprentation

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.15}$$

Consider for example, the observer $A = \sigma_1$ with eignevectors

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right); \qquad A|\uparrow\rangle = |\uparrow\rangle\,, \tag{1.16}$$

$$|\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right); \qquad A|\downarrow\rangle = -|\downarrow\rangle\,. \tag{1.17}$$

A measurement of A for the state $|\uparrow\rangle$ will always yield the result '1' with unit probability whereas the measurement of H for this state will yield $+E$ with probability 1/2 and $-E$ with probability 1/2.

A two state system is sometimes referred to as a *qubit* analogous to a classical binary *bit* of information that can store information being either *on* or *off*. Note however that for a classical bit, a linear superposition of the 'on' and 'off' states is not physically meaningful. In this respect, a qubit is fundamentally different from a classical bit because of the possibility of *coherent* quantum superposition.

1.3 Quantum Coherence

Even though the overall phase of a and b has no physical significance, the *relative* phase is of crucial importance. We will briefly discuss this connection with the information theory later after introducing the notion of entropy.

for a qubit it is meaningful to consider a linear superposition of A qubit is essentially different from a classical bit in that contains information also in the relative phases which is sometimes referred to as *quantum coherence*. It allows for many of the surprising possibilities in *quantum* information theory.

1.4 Density Matrix

Given a state $|\psi\rangle$ one can define the corresponding density matrix

$$\rho(|\psi\rangle) = |\psi\rangle\langle\psi| \tag{1.18}$$

For the state $|\psi\rangle$ of the two-state system given by (1.16)

$$\rho(|\psi\rangle) = |\psi\rangle\langle\psi| = \begin{pmatrix} a^* \\ b^* \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} |a|^2 & a^*b \\ b^*a & |b|^2 \end{pmatrix} . \tag{1.19}$$

This density matrix is by definition self-adjoint, and has unit trace because the state $|\psi\rangle$ is normalized to have unit norm.

For example, the density matrix for the state $|\uparrow\rangle\langle\uparrow|$ is given by

$$\rho(|\downarrow\rangle) := |\downarrow\rangle\langle\downarrow| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.20}$$

whereas the density matrix for the state $|\downarrow\rangle\langle\downarrow|$ is given by

$$\rho(|\downarrow\rangle) := |\downarrow\rangle\langle\downarrow| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{1.21}$$

Note that for both states, a measurement of the observable H yields the result 1 with probability 1/2 and -1 with probability 1/2. This however does not completely characterize the states. The two states are clearly different and they differ from each other by relative phases. We see that the information about the relative phases, or quantum coherence, is contained in the off-diagonal elements of the density matrix.

Given a density matrix one can define its Von-Neumann entropy

$$S = -\mathrm{Tr}_{\mathcal{H}}\rho\log(\rho) . \tag{1.22}$$

For the density matrix $\rho(|\psi\rangle)$ corresponding to any state the Von-Neumann entropy is zero. This can be easily checked by choosing $|\psi\rangle$ to be one of the basis vectors of an orthonormal basis. For example, for our two state system, the density matrix $\rho(|+\rangle)$ takes the form

$$\rho(|1\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \tag{1.23}$$

and hence $S = 0$.

This however suggests a more general notion of the density matrix, with

$$\rho^\dagger = \rho , \quad \mathrm{Tr}\,(\rho) = 1 \tag{1.24}$$

which in a diagonal basis (in an N-dimensional Hilbert space) takes the form

$$\begin{pmatrix} p_1 & 0 & .. & 0 \\ 0 & p_2 & .. & 0 \\ .. & .. & .. & 0 \\ 0 & 0 & 0 & p_N \end{pmatrix}, \qquad \sum_{i=1}^{N} p_i = 1\,. \tag{1.25}$$

A physical interpretation of this density matrix is that the we do not know the state of the system precisely. We know only that the probability that the system is in state $|i\rangle$ is p_i. The The Von Neumann entropy of this density matrix is nonzero and is given by

$$S = -\sum_{i=1} p_i \log p_i \tag{1.26}$$

The density matrix for which the Von Neumann entropy is nonzero is said to correspond to a *mixed* state. By contrast, the density matrix for which the Von Neumann entropy is zero is said to correspond to a *pure* state.

1.5 Quantum Bosonic Oscillator

The hamiltonian of a quantum bosonic[3] oscillator with an angular frequency of oscillation ω is given by

$$H = \frac{\omega}{2}(a^\dagger a + a a^\dagger)\,, \tag{1.27}$$

where a is called the annihilation operator and $a^\dagger$ is called the creation operator. They satisfy the Heisenberg commutation relation

$$[a, a^\dagger] = 1\,, \qquad [a, a] = 0\,, \qquad [a^\dagger, a^\dagger] = 0\,. \tag{1.28}$$

One can define a number operator

$$N = a^\dagger a\,. \tag{1.29}$$

It follows from (1.28) that

$$[N, a] = -a\,, \qquad [N, a^\dagger] = a^\dagger\,. \tag{1.30}$$

[3] The nomenclature of *Bose* and *Fermi* oscillator is explained in §2. Essentially, a *bosonic* oscillator satisfies *commutation* relations (1.28) with Hamiltonian (1.35) whereas a *fermionic* oscillator satisfies *anti-commutation* relations (1.45) with Hamiltonian (1.44).

Thus, a lowers the N eigenvalue by one whereas $a^\dagger$ raises it by one. We seek a unitary representation of the Heisenberg algebra (1.28). For every state $|\psi\rangle$ in a unitary representation,

$$\langle\psi|N|\psi\rangle = \psi|a^\dagger a|\psi\rangle = |a|\psi\rangle|^2 \geq 0\,. \tag{1.31}$$

Moreover, $N|\psi\rangle = 0$ iff $a|\psi\rangle = 0$. Let us denote the null eigenvector of N by $|0\rangle$. This state is often referred to as the *Fock vacuum* of the system. One can then construct the *Fock representation* of the Heisenberg algebra generated by states $\{|n\rangle\}$, where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,, \qquad n = 0, 1, 2. \ldots\,. \tag{1.32}$$

It is easy to see that

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{1.33}$$

$$N|n\rangle = n|n\rangle\,. \tag{1.34}$$

It is clear that the Hamiltonian is related to the number operator by

$$H = \omega(N + \frac{1}{2}) = \omega(a^\dagger a + \frac{1}{2})\,. \tag{1.35}$$

The Heisenberg equations of motion for the creation and annihilation operators are

$$i\frac{da}{dt} = -[H, a] = \omega a\,, \qquad i\frac{da^\dagger}{dt} = -[H, a^\dagger] = -\omega a^\dagger \tag{1.36}$$

These first order equations can be easily solved to obtain

$$a(t) = a(0)e^{-i\omega t}\,, \qquad a^\dagger(t) = a^\dagger(0)e^{i\omega t}\,. \tag{1.37}$$

Note that a and $a^\dagger$ are not self-adjoint operators but rather are adjoints of each other. One can define self-adjoint operators Q and P by[4]

$$\begin{aligned} a &= \sqrt{\tfrac{w}{2}}\left(Q + \tfrac{i}{w}P\right)\,, \qquad & a^\dagger &= \sqrt{\frac{w}{2}}\left(Q - \frac{i}{w}P\right)\,; \\ Q &= \sqrt{\tfrac{1}{2w}}\left(a + a^\dagger\right)\,, \qquad & P &= \sqrt{\frac{w}{2i}}\left(a - a^\dagger\right)\,. \end{aligned} \tag{1.38}$$

[4]The factors of w are added so that Q and P can be identified with the 'position' and 'momentum' of a mechanical oscillator of unit mass such as a ball attached to a spring with spring constant equal to w^2. In our discussions the oscillator will refer always to the harmonic mode of a quantum field and not to a mechanical oscillator.

The 'position' and 'momentum' satisfy the Heisenberg commutation relation

$$[Q, P] = i\,. \tag{1.39}$$

The Hamiltonian can then be written as

$$H = \frac{1}{2}\left(P^2 + w^2 Q^2\right)\,. \tag{1.40}$$

The observable Q and P satisfy the Heisenberg equations of motion

$$\frac{dQ}{dt} = P\,, \tag{1.41}$$

$$\frac{dP}{dt} = -\omega^2 Q \tag{1.42}$$

which are first order differential equations in time derivative. For a mechanical oscillator, the operator Q can be thought of as the 'coordinate' and P is called the 'momentum conjugate to Q' which is given simply by the first time derivative of Q. One can eliminate P to obtain the equation of motion for Q alone:

$$\frac{d^2Q}{dt^2} = -\omega^2 Q \tag{1.43}$$

which is a second order differential equation in time derivative.

1.6 Quantum Fermionic Oscillator

The hamiltonian of a quantum fermionic oscillator with an angular frequency of oscillation ω is given by

$$H = \frac{\omega}{2}(b^\dagger b - b b^\dagger)\,, \tag{1.44}$$

where b is called the fermionic annihilation operator and $b^\dagger$ is called the fermionic creation operator. They satisfy the Heisenberg *anti*-commutation relation

$$\{b, b^\dagger\} = 1\,, \qquad \{b, b\} = 0\,, \qquad \{b^\dagger, b^\dagger\} = 0 \tag{1.45}$$

where the anti-commutator of two 'fermionic' operators A and B is defined by

$$\{A, B\} := AB + BA \tag{1.46}$$

One can define a fermion number operator as before by

$$F = b^\dagger b\,. \tag{1.47}$$

It follows from (1.28) that

$$[F, b] = -b\,, \qquad [F, b^\dagger] = b^\dagger\,. \tag{1.48}$$

The Hamiltonian is given by

$$H = \omega(F - \frac{1}{2}) = \omega(b^\dagger b - \frac{1}{2})\,. \tag{1.49}$$

The unitary Fock representation of the Heisenberg algebra (1.45) is now particularly simple. As before, one can define the Fock vacuum as the state annihilated by N and hence by b. However, unlike for the bosonic oscillator, $(b^\dagger)^2 = 0$, as a result the Fock representation terminates and we have only a two state representation:

$$|0\rangle\,, \qquad |1\rangle = b^\dagger|0\rangle\,, \tag{1.50}$$

such that

$$b|0\rangle = 0\,, \qquad b|1\rangle = |0\rangle\,; \tag{1.51}$$

$$b^\dagger|0\rangle = |1\rangle\,, \qquad b^\dagger|1\rangle = 0\,. \tag{1.52}$$

Various operators thus have a simple matrix representation

$$b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{1.53}$$

$$N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H = \frac{\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.54}$$

For the fermionic oscillator, there is a natural $\mathbb{Z}_2$ grading defined by the operator $(-1)^F$ which squares to one. Thus, operators that commute with $(-1)^F$ (such as F and H) are *even* or bosonic, whereas operators that anticommute with $(-1)^F$ (such as b and $b^\dagger$) are *odd* or fermionic. If one assigns fermion number 0 to the Fock vacuum, then $|0\rangle$ is even whereas $|1\rangle$ is odd.

1.7 Partition Functions

Given a quantum system with a Hilbert space $\mathcal{H}$ and Hamiltonian H, one can define its *partition function* as the trace

$$Z(q) = \mathrm{Tr}_{\mathcal{H}}\left[q^H\right]\,. \tag{1.55}$$

The partition function can be viewed as a character of the Hamiltonian. As we will see in the next section, this mathematical object is of fundamental physical significance.

1.7.1 Bosonic Oscillator

It is easy to compute the partition function for the quantum bosonic oscillator. The eigenvalues of the Hamiltonian have energy $\epsilon_n = \omega(n + \frac{1}{2})$, $(n = 0, 1, \ldots)$ and the the eigenvectors $\{|n\rangle\}$ form a complete basis. Hence the partition function is given by

$$\begin{aligned} Z(q) &= \sum_{n=0}^{\infty} q^{w(n+\frac{1}{2})} && (1.56)\\ &= q^{\frac{\omega}{2}} \left(1 + q^{\omega} + q^{2\omega} + q^{3\omega} + \ldots\right) && (1.57)\\ &= \frac{q^{\frac{\omega}{2}}}{1 - q^{\omega}} . && (1.58)\end{aligned}$$

1.7.2 Fermionic Oscillator

One can similarly define the partition function for the fermionic oscillator. The eigenvalues of the Hamiltonian have energy $\epsilon_n = \omega(n - \frac{1}{2})$, $(n = 0, 1)$ and the the eigenvectors $\{|n\rangle\}$ form a complete basis. Hence the partition function is given by

$$\begin{aligned} Z(q) &= \sum_{n=0}^{1} q^{w(n-\frac{1}{2})} && (1.59)\\ &= q^{-\frac{\omega}{2}} \left(1 + q^{\omega}\right) . && (1.60)\end{aligned}$$

For the fermionic oscillator, one can also compute the *indexed* partition function including the $\mathbb{Z}_2$ grading in the trace:

$$Z(q) = \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F \; q^H\right] . \tag{1.61}$$

It can be readily evaluated:

$$Z(q) = q^{-\frac{\omega}{2}} \left(1 - q^{\omega}\right) . \tag{1.62}$$

1.7.3 Multiple oscillators

One can also consider a system consisting of R bosonic oscillators of frequencies $\{\omega_1, \omega_2, \ldots \omega_R\}$ and associated Hilbert spaces $\{\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_R\}$. In this case, the total Hilbert space is the product space

$$\mathcal{H} = \bigotimes_{r=1}^{R} \mathcal{H}_r . \tag{1.63}$$

The total Hamiltonian is the sum of the individual Hamiltonians:

$$H = \sum_{r=1}^{R} H_r \tag{1.64}$$

$$= \sum_{r=1}^{R} \omega_r a_r^\dagger a_r \,. \tag{1.65}$$

A typical eigenstate of the Hamiltonian is of the form

$$|\psi\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \ldots \otimes |n_R\rangle \,, \tag{1.66}$$

with the energy eigenvalue (= total energy of the system)

$$E = \sum_{r=1}^{R} n_r \omega_r + E_0 \,. \tag{1.67}$$

where we have defined E_0 as the energy of the Fock vacuum

$$E_0 := \frac{1}{2} \sum_{r=1}^{R} w_r \tag{1.68}$$

The partition function of the combined system of I oscillators is then a product of individual partition functions:

$$Z(q) = q^{E_0} \prod_{r=1}^{R} \frac{1}{1 - q^{\omega_r}} \,. \tag{1.69}$$

Consider now a system of infinite oscillators so that $R \to \infty$ with *integer* frequencies so that $w_r = r$ $(r = 1, 2, \ldots, \infty)$. In this case, we have

$$H = \sum_{r=1}^{\infty} r \left(a_r^\dagger a_r + \frac{1}{2} \right) \,, \tag{1.70}$$

and the partition function is given by

$$Z(q) = q^{E_0} \prod_{r=1}^{\infty} \frac{1}{(1 - q^r)} \,. \tag{1.71}$$

The ground state energy in this case is a divergent sum. To obtain a sensible finite answer one can first define the *regularized* ground state energy E_0^s by using ζ-function regularization:

$$E_0^s = \frac{1}{2} \sum_r r^{-s} = \frac{1}{2} \zeta(s) \,. \tag{1.72}$$

After evaluating the sum in terms of the Riemann ζ function $\zeta(s)$, and using $\zeta(-1) = -\frac{1}{12}$, one can define a finite[5] *renormalized* ground state energy by

$$E_0 = \frac{1}{2} \lim_{s \to -1} \zeta(s) \tag{1.73}$$

With this renormalization, the partition function for the system of infinite oscillators of integer frequencies is given by

$$Z(q) = q^{-\frac{1}{24}} \prod_{r=1}^{\infty} \frac{1}{(1 - q^r)} \,. \tag{1.74}$$

which we recognize as the Dedekind $\eta(\tau)$ function.

1.8 Quantum Ensemble and Entropy

In many physical situations, one may not have complete knowledge about the state of the system under consideration. For example, for the air in this room, one may not have precise information about the states of all the air molecules but one may know only that the total energy is E. In this case, one is naturally led to the notion of an *ensemble* of quantum states.

An ensemble of fixed energy E is simply the eigen-subspace $\mathcal{H}(E)$ of the Hamiltonian H with eigenvalue E. The *quantum degeneracy* $d(E)$ is is the multiplicity of the energy eigenvalue E

$$d(E) := \dim\left(\mathcal{H}(E)\right) \,. \tag{1.75}$$

In physics literature $d(E)$ is sometimes referred to as the the total number of *microstates* of the system with total energy E.

The entropy $S(E)$ of this ensemble by the Boltzmann relation

$$S(E) = \log d(E) \,. \tag{1.76}$$

The logarithm ensures that the total entropy of a system consisting of identical subsystems is *additive* much like the total energy. Thus, for N subsystems each with energy E, the total energy is NE, and the ensemble of interest is the eigensubspace of the total Hamiltonian

$$\mathcal{H}_{tot}(IE) = \mathcal{H}(E) \otimes \mathcal{H}(E) \ldots \otimes \mathcal{H}(E) \tag{1.77}$$

[5] We discuss the physical justification for the *renormalization* procedure in more detail in §2.4 in the context of local quantum field theories. Similar procedure was used by Euler.

with dimension

$$d_{tot}(NE) = d(E)^N \tag{1.78}$$

and hence the total entropy is $S_{tot}(NE) = NS(E)$.

The micro-canonical density matrix for an isolated system of energy E is thus proportional to the identity matrix in the Hilbert subspace $\mathcal{H}(E)$:

$$\rho(E) = \frac{1}{d(E)}\mathbf{1} \tag{1.79}$$

We see that the Von-Neumann entropy of this density matrix equals the entropy defined above:

$$\mathrm{Tr}_{\mathcal{H}(E)}\left(\rho\log(\rho)\right) = d(E) = S(E)\,. \tag{1.80}$$

The study of ensembles and their physical consequences is an important and profound branch of physics known as *Quantum Statistical Mechanics.* Many of the secrets of the quantum theory of matter were deduced indirectly well before the final formulation of the theory in the 20th century. This was possible through a deft use of statistical reasoning by some of the masters of early statistical mechanics such as Boltzmann, Maxwell, Gibbs, and Einstein.

of the theory through Boltzmann relation

$$S(E, Q, J) = \log(d(E, Q, J),$$

One may know only a few conserved quantum numbers[6] of the system such as the total energy E or the total charge Q or the total spin J of the system.

where d is the the degeneracy or the total number of microstates of the system of for a given energy.

1.9 Canonical Ensemble and Temperature

The right-hand side of the Boltzmann relation is defined in terms of the *microscopic* properties of the system, namely, the dimension of the Hilbert eigensubspace and as it it stands it is simply a definition. The fundamental significance of entropy stems from the fact the left hand side of the Boltzmann relation has an independent definition in terms of *macroscopic* thermodynamic properties. Thus, the Boltzmann relation provides a link between the

[6]Quantum number of a conserved observable translates according to our dictionary to an eigenvalue of a self-adjoint operator that commutes with the Hamiltonian.

micro and macro world. The gross thermodynamic properties of a system can be used to deduce nontrivial information about the *microscopic* structure. Entropy is not a kinematic quantity like energy or momentum but rather contains information about the total number microscopic degrees of freedom of the system. Because of the Boltzmann relation, one can learn a great deal about the microscopic properties of a system from its thermodynamics properties.

Much of the power of statistical mechanics through the notion of entropy and a fundamental relation due to

This is the famous 'Bose-Einstein' distribution. The Planck distribution law follows.

This is the famous 'Fermi-Dirac' distribution.

1.10 Entropy, Disorder, and Information

The total information that can be stored in a system with a density matrix ρ on a d-dimensional vector space d can thus be defined by

$$I = I_{max} - S \tag{1.81}$$

Chapter 2

Quantum Fields and Particles

All physical processes take place in space and time. An *event* in d-dimensional space can be specified by its spatial coordinates $(x^1, x^2, \ldots, x^d)$ and the time t of its occurrence. In Newtonian physics, time is *absolute*, in that all observers in relative motion with respect to each other record the same time. In Einsteinian physics, time is *relative*, in that different observers in relative motion experience different times and in general time coordinate can 'mix' with the spatial coordinates. This *relativity of time* implies that one must regard *spacetime* as single entity combining space and time together.

We start with a few definitions about the spacetime and classical fields and then explain how one can define quantum fields using the quantum oscillators introduced earlier.

2.1 Spacetime Manifold and Classical Fields

A $(1 + d)$-dimensional spacetime is a differential manifold $M^{1,d}$ with local coordinates $\{x^\mu\}$, $(\mu = 0, 1, 2, \ldots d)$ where the coordiate x^0 is interpreted as the time coordinate t and the coordinates x^i, $(i = 1, 2, \ldots d)$ are the spatial coordinates. In Einsteins General Theory of Relativity, $M^{1,d}$ is *pseudo*-Riemannian manifold equipped with a metric $g_{\mu\nu}$ with *Lorentzian* signature[1] $- + \ldots +$. Recall that the metric is rank-2, symmetric, covariant, tensor field which defines the local line element by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\, . \tag{2.1}$$

The simplest example of a spacetime is the Minkowski spacetime $\mathbb{R}^{1,d}$

[1] By contrast, the metric for a Riemannian manifold has Euclidean signature $+ + \ldots +$.

where the metric is the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & .. & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

and the local line element

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = -(dx^0)^2 + (dx^1)^2 + \ldots + (dx^d)^2\,. \tag{2.3}$$

Thus, a Minkowski spacetime is just a Lorentzian analog of Euclidean space $\mathbb{R}^{1+d}$ with the Euclidean metric $\delta_{\mu\nu}$.

The structure group of the tangent space is $SO(1,d)$ and there is a natural Minkowski metric defined on the tangent space. One can thus define a tangent bundle on M. In physics, one considers the double cover $Spin(1,d)$ which admits spinor representations. This allows one to define the spinor bundle. One can also consider a more general associated vector bundle with the $Spin(1,d) \times G$ where G is a compact Lie group. Various *classical fields* that occur in physics can be thought of as sections of these bundles.

The best known example of a classical field is the electromagnetic field introduced by Faraday and Maxwell. There are several other fields of importance in physics such as the scalar field sometimes referred to as Klein-Gordon field, the Dirac spinor field and the metric tensor field.

On a Riemannian manifold one can define the Christoffel symbols $\Gamma_{\alpha\mu\nu}$ determined in terms of the first derivatives of the metric

$$\Gamma_{\alpha\mu\nu} \tag{2.4}$$

The Christoffel symbols determine the Levi-Civita connection which can be used to define parallel transport and covariant derivatives for the tensor fields. Each of the classical fields satisfy an *equation of motion* which is a covariant partial differential equation similar to the Laplace equation on a Euclidean manifold. We will see in the next section that one can associate a quantum field with a given classical field. The classical field equation can then be viewed as the classical version of the the Heisenberg equation of motion for the corresponding quantum field.

- **Scalar field φ:**

  The simplest example of a classical field is a real scalar field $\varphi(x)$ which is scalar-valued function on $M^{1,d}$ or equivalently a scalar-valued map from $M^{1,d}$ to $\mathbb{R}$. A massless free scalar field satisfies a second-order differential equation known as the massless Klein-Gordon equation

$$\Delta_g \varphi = 0 \tag{2.5}$$

where Δ_g is the Lorentzian analog of the scalar Laplacian, which is sometime referred to as the d'Alembertian. It is defined by

$$\Delta_g := d^* d \tag{2.6}$$

where d is the de-Rham operator and * is the Hodge star operation.

- **Vector field** $A_\mu(x)$

  A (covariant) vector field (or equivalently the connectoin one-form) $A := A_\mu dx^\mu$ corresponds to the electromagnetic potential. The one form A can be thought of a section of the principle bundle with structure group $U(1)$. One can define the field strength (or equivalently the curvature two-form) by $F := dA$ with the Bianchi identity $dF = 0$. The free vector field satisfies the equation of motion

$$d^* F = 0\,. \tag{2.7}$$

  All the rich physics of classical maxwell electrodynamics such as the propagation of electromagnetic waves follows from these equations of motion. More generally, one can consider a principle bundle with structure group G. The section of this bundle is called the *gauge field* with *gauge group* G.

- **Spinor field** $\Psi_\alpha(x)$:

  The Dirac spinor field is a section of the spin bundle on the manifold. A spinor field satisfies a first order differential equation known as the Dirac equation

$$\Gamma^\mu D_\mu \Psi = 0 \tag{2.8}$$

2.2 A Scalar field in 1+1 Dimensions

Consider the Hamilotonian of a system of infinite number of oscillators with integral frequencies $1, 2, 3, \ldots$ that we considered earlier:

$$H = \sum_{r=1}^{\infty} r \left(a_r^\dagger a_r + \frac{1}{2} \right) \tag{2.9}$$

which we will refer to as the left-moving Hamiltonian. Consider another identical copy

$$\tilde{H} = \sum_{\tilde{r}=1}^{\infty} \left(\tilde{r}\tilde{a}_r^\dagger \tilde{a}_r + \frac{1}{2} \right) \tag{2.10}$$

which we will refer to as a right-moving Hamiltonian. Now, consider a Lorentzian Manifold $M^{1,1}$ with coordinates (t, x). We further assume that the manifold is topologically a cylinder with the identifications

$$(t, x) \sim (t, x + 2\pi)\,. \tag{2.11}$$

Given this set of oscillators, one can define and operator-valued *left-moving field* $\varphi_L(x, t)$ by

$$\varphi_L(t - x) := \sum_{r=1}^{\infty} \frac{1}{\sqrt{2r}} \left[a_r e^{-ir(t-x)} + a_r^\dagger e^{+ir(t-x)} \right] \tag{2.12}$$

and similarly a *right-moving* field

$$\varphi_R(t + x) := \sum_{r}^{\infty} \frac{1}{\sqrt{2r}} \left[\tilde{a}_r e^{-ir(t-x)} + \tilde{a}_r^\dagger e^{+ir(t-x)} \right] \tag{2.13}$$

Together, one can define a scalar quantum field on $\mathbb{R}^{1,1}$

$$\varphi(t, x) = \varphi_L(t - x) + \varphi_R(t + x) + \phi_0 \tag{2.14}$$

Thus a *quantum field* can be thought of as a operator valued function which is the quantum version of a classical field which is a scalar valued function. More generally, a quantum field $\varphi(t, x)$ is an operator-valued *distribution*.

Note that this field satisfies the massless Klein-Gordon equation in $1 + 1$ dimensions with flat metric $g = \eta$:

$$\Delta_\eta \varphi = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) \varphi = 0 \tag{2.15}$$

This can be seen easily by defining the cooordinates

$$x^+ := t + x\,, \qquad x^- := t - x\,, \tag{2.16}$$

so that

$$\Delta_\eta \varphi = 4 \frac{\partial^2}{\partial x^+ \partial x^+} \varphi \tag{2.17}$$

which vanishes trivially because φ_L is independent of x^+ whereas φ_R is independent of x^-.

We can rewrite the quantum field as

$$\varphi(t,x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2r}} \left[a_r e^{-ir(t-x)} + a_r^\dagger e^{+ir(t-x)}\right] \tag{2.18}$$

If the equations of motion of the quantum field is Weyl invariant then it is called a conformal field. In this case the quantum field theory is called a conformal field theory.

2.3 A Dirac field in 1+1 Dimensions

2.4 Renormalization

Let us consider the ground state energy of the left-moving oscillators:

$$E_0 = \frac{1}{2}\sum_{r=1}^{\infty} r \tag{2.19}$$

It is clearly divergent but can be 'regularized', for example as

$$E_0^\epsilon = \frac{1}{2}\sum_{r=1}^{\infty} r e^{-\epsilon r} \tag{2.20}$$

for a small ϵ. This can be readily summed by noting that

$$E_0^\epsilon = -\frac{1}{2}\frac{\partial}{\partial\epsilon}\sum_{r=1}^{\infty} e^{-\epsilon r} = -\frac{1}{2}\frac{\partial}{\partial\epsilon}\left(\frac{1}{1-e^{-\epsilon}}\right) \tag{2.21}$$

$$= \frac{e^\epsilon}{2(e^\epsilon - 1)^2} \tag{2.22}$$

We have introduced a quantum field is a collection of infinite number of quantum oscillators. The Hilbert space is infinite product and the Hamilotnian is an infinite sum. Consequently, there is a potential for divergences. These divergences can be studying a regularized theory by putting a cutoff which renders the infinite sums finite and well-defined. Renormalization theory is the study of this limit. More generally, when interactions are present

2.5 A Quantum Field in 1+ 3 Dimensions

The Planck distribution law thus follows

2.6 A Macro-window into the Micro-world

This example illustrates how For example the number density of photons, which is closely related to the entropy density, scales with temperature as T^3 whereas the energy density scales as T^4. This is a concrete *macroscopic* prediction of the highly abstract *microscopic* model of the photon gas in terms of the quantum oscillators associated with the electromagnetic field. Experimental verification of this prediction provides a macroscopic confirmation of the microscopic model. It is in this way that statistical reasoning together with the Boltzmann relation provides a macroscopic window into the microscopic world.

2.7 Elementary Particles

2.8 Symmetries

2.9 Interactions

Chapter 3

Classical Black Holes

3.1 General Relativity and Classical Gravity

Curvature of spacetime is gravity and the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ which is determined in terms of the second derivatives of the metric $\partial_\alpha \partial_\beta g_{\mu\nu}$

Matter dictates how spacetime curves. Spacetime dictates how matter moves.

To understand the relevant parameters and the geometry of black holes, let us first consider the Einstein-Maxwell theory described by the action

$$\frac{1}{16\pi G}\int R\sqrt{g}d^4x - \frac{1}{16\pi}\int F^2\sqrt{g}d^4x, \tag{3.1}$$

where G is Newton's constant, $F_{\mu\nu}$ is the electro-magnetic field strength, R is the Ricci scalar of the metric $g_{\mu\nu}$. In our conventions, the indices μ, ν take values $0, 1, 2, 3$ and the metric has signature $(-, +, +, +)$.

3.2 Schwarzschild Black Hole

Consider the Schwarzschild metric which is a spherically symmetric, static solution of the vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ that follow from (3.1) when no electromagnetic fields are excited. This metric is expected to describe the spacetime outside a gravitationally collapsed non-spinning star with zero charge. The solution for the line element is given by

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where t is the time, r is the radial coordinate, and Ω is the solid angle on a 2-sphere. This metric appears to be singular at $r = 2GM$ because some of its components vanish or diverge, $g_{00} \to \infty$ and $g_{rr} \to \infty$. As is well known, this is not a real singularity. This is because the gravitational tidal forces are finite or in other words, components of Riemann tensor are finite in orthonormal coordinates. To better understand the nature of this apparent singularity, let us examine the geometry more closely near $r = 2GM$. The surface $r = 2GM$ is called the 'event horizon' of the Schwarzschild solution. Much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon.

To focus on the near horizon geometry in the region $(r - 2GM) \ll 2GM$, let us define $(r - 2GM) = \xi$, so that when $r \to 2GM$ we have $\xi \to 0$. The metric then takes the form

$$ds^2 = -\frac{\xi}{2GM}dt^2 + \frac{2GM}{\xi}(d\xi)^2 + (2GM)^2 d\Omega^2, \tag{3.2}$$

up to corrections that are of order $(\frac{1}{2GM})$. Introducing a new coordinate ρ,

$$\rho^2 = (8GM)\xi \quad \text{so that} \quad d\xi^2 \frac{2GM}{\xi} = d\rho^2,$$

the metric takes the form

$$ds^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2 + (2GM)^2 d\Omega^2. \tag{3.3}$$

From the form of the metric it is clear that ρ measures the geodesic radial distance. Note that the geometry factorizes. One factor is a 2-sphere of radius $2GM$ and the other is the (ρ, t) space

$$ds_2^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2. \tag{3.4}$$

We now show that this $1 + 1$ dimensional spacetime is just a flat Minkowski space written in funny coordinates called the Rindler coordinates.

3.3 Rindler coordinates

To understand Rindler coordinates and their relation to the near horizon geometry of the black hole, let us start with $1 + 1$ Minkowski space with the usual flat Minkowski metric,

$$ds^2 = -dT^2 + dX^2. \tag{3.5}$$

In light-cone coordinates,

$$U = (T + X) \quad V = (T - X), \tag{3.6}$$

the line element takes the form

$$ds^2 = -dU\ dV. \tag{3.7}$$

Now we make a coordinate change

$$U = \frac{1}{\kappa}e^{\kappa u}, \quad V = -\frac{1}{\kappa}e^{-\kappa v}, \tag{3.8}$$

to introduce the Rindler coordinates (u, v). In these coordinates the line element takes the form

$$ds^2 = -dU\ dV = -e^{\kappa(u-v)}du\ dv. \tag{3.9}$$

Using further coordinate changes

$$u = (t + x), \quad v = (t - x), \quad \rho = \frac{1}{\kappa}e^{\kappa x}, \tag{3.10}$$

we can write the line element as

$$ds^2 = e^{2\kappa x}(-dt^2 + dx^2) = -\rho^2\kappa^2 dt^2 + d\rho^2. \tag{3.11}$$

Comparing (3.4) with this Rindler metric, we see that the (ρ, t) factor of the Schwarzschild solution near $r \sim 2GM$ looks precisely like Rindler spacetime with metric

$$ds^2 = -\rho^2\kappa^2\ dt^2 + d\rho^2 \tag{3.12}$$

with the identification

$$\kappa = \frac{1}{4GM}.$$

This parameter κ is called the surface gravity of the black hole. For the Schwarzschild solution, one can think of it heuristically as the Newtonian acceleration GM/r_H^2 at the horizon radius $r_H = 2GM$. Both these parameters– the surface gravity κ and the horizon radius r_H play an important role in the thermodynamics of black hole.

This analysis demonstrates that the Schwarzschild spacetime near $r = 2GM$ is not singular at all. After all it looks exactly like flat Minkowski space times a sphere of radius $2GM$. So the curvatures are inverse powers of the radius of curvature $2GM$ and hence are small for large $2GM$.

3.4 Exercises

Uniformly accelerated observer and Rindler coordinates

Consider an astronaut in a spaceship moving with constant acceleration a in Minkowski spactime with Minkowski coordinates $(T, \vec{X})$. This means she feels a constant normal reacting from the floor of the spaceship in her rest frame:

$$\frac{d^2\vec{X}}{dt^2} = \vec{a}\,, \quad \frac{dT}{d\tau} = 1 \tag{3.13}$$

where τ is proper time and $\vec{a}$ is the acceleration 3-vector.

1. *Write the equation of motion in a covariant form and show that her 4-velocity* $u^\mu := \frac{dX^\mu}{d\tau}$ *is timelike whereas her 4-acceleration* a^μ *is spacelike.*
2. *Show that if she is moving along the* x *direction, then her trajectory is of the form*

$$T = \frac{1}{a}\sinh(a\tau)\,, \quad X = \frac{1}{a}\cosh(a\tau) \tag{3.14}$$

   which is a hyperboloid. Find the acceleration 4-vector.
3. *Show that it is natural for her to use her proper time as the time coordinate and introduce a coordinate frame of a family of observers with*

$$T = \zeta\,\sinh(a\eta)\,, \quad X = \zeta\,\cosh(a\eta)\,. \tag{3.15}$$

By examining the metric, show that $v = \eta - \zeta$ and $u = \eta + \zeta$ are precisely the Rindler coordinates introduced earlier with the acceleration parameter a identified with the surface gravity κ.

3.5 Kruskal extension

One important fact to note about the Rindler metric is that the coordinates u, v do not cover all of Minkowski space because even when the vary over the full range

$$-\infty \leq u \leq \infty, \quad -\infty \leq v \leq \infty$$

the Minkowski coordinate vary only over the quadrant

$$0 \leq U \leq \infty, \quad -\infty < V \leq 0. \tag{3.16}$$

If we had written the flat metric in these 'bad', 'Rindler-like' coordinates, we would find a fake singularity at $\rho = 0$ where the metric appears to become singular. But we can discover the 'good', Minkowski-like coordinates U and V and extend them to run from $-\infty$ to ∞ to see the entire spacetime.

Since the Schwarzschild solution in the usual (r, t) Schwarzschild coordinates near $r = 2GM$ looks exactly like Minkowski space in Rindler coordinates, it suggests that we must extend it in properly chosen 'good' coordinates. As we have seen, the 'good' coordinates near $r = 2GM$ are related to the Schwarzschild coordinates in exactly the same way as the Minkowski coordinates are related the Rindler coordinates.

In fact one can choose 'good' coordinates over the entire Schwarzschild spacetime. These 'good' coordinates are called the Kruskal coordinates. To obtain the Kruskal coordinates, first introduce the 'tortoise coordinate'

$$r^* = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right). \tag{3.17}$$

In the (r^*, t) coordinates, the metric is conformally flat, *i.e.*, flat up to rescaling

$$ds^2 = (1 - \frac{2GM}{r})(-dt^2 + dr^{*2}). \tag{3.18}$$

Near the horizon the coordinate r^* is similar to the coordinate x in (3.11) and hence $u = t + r^*$ and $v = t - r^*$ are like the Rindler (u, v) coordinates. This suggests that we define U, V coordinates as in (3.8) with $\kappa = 1/4GM$. In these coordinates the metric takes the form

$$ds^2 = -e^{-(u-v)\kappa} dU\ dV = -\frac{2GM}{r} e^{-r/2GM} dU\ dV \tag{3.19}$$

We now see that the Schwarzschild coordinates cover only a part of spacetime because they cover only a part of the range of the Kruskal coordinates. To see the entire spacetime, we must extend the Kruskal coordinates to run from $-\infty$ to ∞. This extension of the Schwarzschild solution is known as the Kruskal extension.

Note that now the metric is perfectly regular at $r = 2GM$ which is the surface $UV = 0$ and there is no singularity there. There is, however, a real singularity at $r = 0$ which cannot be removed by a coordinate change because physical tidal forces become infinite. Spacetime stops at $r = 0$ and at present we do not know how to describe physics near this region.

3.6 Event horizon

We have seen that $r = 2GM$ is not a real singularity but a mere coordinate singularity which can be removed by a proper choice of coordinates.

Thus, locally there is nothing special about the surface $r = 2GM$. However, globally, in terms of the causal structure of spacetime, it is a special surface and is called the 'event horizon'. An event horizon is a boundary of region in spacetime from behind which no causal signals can reach the observers sitting far away at infinity.

To see the causal structure of the event horizon, note that in the metric (3.11) near the horizon, the constant radius surfaces are determined by

$$\rho^2 = \frac{1}{\kappa^2} e^{2\kappa x} = \frac{1}{\kappa^2} e^{\kappa u} e^{-\kappa v} = -UV = \text{constant} \tag{3.20}$$

These surfaces are thus hyperbolas. The Schwarzschild metric is such that at $r \gg 2GM$ and observer who wants to remain at a fixed radial distance $r =$ constant is almost like an inertial, freely falling observers in flat space. Her trajectory is time-like and is a straight line going upwards on a space-time diagram. Near $r = 2GM$, on the other hand, the constant r lines are hyperbolas which are the trajectories of observers in uniform acceleration.

To understand the trajectories of observers at radius $r > 2GM$, note that to stay at a fixed radial distance r from a black hole, the observer must boost the rockets to overcome gravity. Far away, the required acceleration is negligible and the observers are almost freely falling. But near $r = 2GM$ the acceleration is substantial and the observers are not freely falling. In fact at $r = 2GM$, these trajectories are light like. This means that a fiducial observer who wishes to stay at $r = 2GM$ has to move at the speed of light with respect to the freely falling observer. This can be achieved only with infinitely large acceleration. This unphysical acceleration is the origin of the coordinate singularity of the Schwarzschild coordinate system.

In summary, the surface defined by $r =$ contant is timelike for $r > 2GM$, spacelike for $r < 2GM$, and light-like or null at $r = 2GM$.

In Kruskal coordinates, at $r = 2GM$, we have $UV = 0$ which can be satisfied in two ways. Either $V = 0$, which defines the 'future event horizon', or $U = 0$, which defines the 'past event horizon'. The future event horizon is a one-way surface that signals can be sent into but cannot come out of. The region bounded by the event horizon is then a black hole. It is literally a hole in spacetime which is black because no light can come out of it. Heuristically, a black hole is black because even light cannot escape its strong gravitation pull. Our analysis of the metric makes this notion more precise. Once an observer falls inside the black hole she can never come out because to do so she will have to travel faster than the speed of light.

As we have noted already $r = 0$ is a real singularity that is inside the event horizon. Since it is a spacelike surface, once a observer falls insider the event horizon, she is sure to meet the singularity at $r = 0$ sometime in future no matter how much she boosts the rockets.

In our example of the Schwarzschild black hole, the event horizon is static because it is defined as a constant r hypersurface $r = 2GM$ which does not change with time. More precisely, the time-like Killing vector $\frac{\partial}{\partial t}$ leaves it invariant. It is at the same time null because g^{rr} vanishes at $r = 2GM$ so that the norm of the 1-form dr vanishes. In general, as for a spinning Kerr-Newman black hole, the horizon is not static but only stationary (because of the uniform rotation) and null.

In summary, an event horizon is a surface that is simultaneously *stationary* and *null*, which causally separates the inside and the outside of a black hole. For a discussion of the notion of an event horizon in greater generality see [4, 5].

3.7 Black hole parameters

From our discussion of the Schwarzschild black hole we are ready to abstract some important general concepts that are useful in describing the physics of more general black holes.

To begin with, a *black hole* is an asymptotically flat spacetime that contains a region which is not in the backward lightcone of future timelike infinity. The boundary of such a region is a stationary null surface call the *event horizon*. The fixed t slice of the event horizon is a two sphere.

There are a number of important parameters of the black hole. We have introduced these in the context of Schwarzschild black holes. For a general black holes their actual values are different but for all black holes, these parameters govern the thermodynamics of black holes.

1. The radius of the event horizon r_H is the radius of the two sphere. For a Schwarzschild black hole, we have $r_H = 2GM$.

2. The area of the event horizon A_H is given by $4\pi r_H^2$. For a Schwarzschild black hole, we have $A_H = 16\pi G^2 M^2$.

3. The surface gravity is the parameter κ that we encountered earlier. As we have seen, for a Schwarzschild black hole, $\kappa = 1/4GM$.

3.8 Charged Black Hole

The most general static, spherically symmetric, charged solution of the Einstein-Maxwell theory (3.1) gives the Reissner-Nordström (RN) black hole. In what follows we choose units so that $G = \hbar = 1$. The line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.21}$$

and the electromagnetic field strength by

$$F_{tr} = Q/r^2.$$

The parameter Q is the charge of the black hole and M is the mass. For $Q = 0$ this reduces to the Schwarzschild black hole.

From the metric (3.21) we see that the event horizon for this solution is located at where $g^{rr} = 0$, or

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0.$$

Since this is a quadratic equation in r,

$$r^2 - 2QMr + Q^2 = 0,$$

it has two solutions.

$$r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$.

3.9 Historical aside

Apart from its physical significance, the entropy of a black hole makes for a fascinating study in the history of science. It is one of the very rare examples where a scientific idea has gestated and evolved over several decades into an important conceptual and quantitative tool almost entirely on the strength of theoretical considerations. That we can proceed so far with any confidence at all with very little guidance from experiment is indicative of the robustness of the basic tenets of physics. It is therefore worthwhile to place black holes and their entropy in a broader context before coming to the more recent results pertaining to the quantum aspects of black holes within string theory.

A black hole is now so much a part of our vocabulary that it can be difficult to appreciate the initial intellectual opposition to the idea of 'gravitational collapse' of a star and of a 'black hole' of nothingness in spacetime by several leading physicists, including Einstein himself.

To quote the relativist Werner Israel ,

" *There is a curious parallel between the histories of black holes and continental drift. Evidence for both was already non-ignorable by 1916, but both ideas were stopped in their tracks for half a century by a resistance bordering on the irrational.*"

On January 16, 1916, barely two months after Einstein had published the final form of his field equations for gravitation [12], he presented a paper to the Prussian Academy on behalf of Karl Schwarzschild [13], who was then fighting a war on the Russian front. Schwarzschild had found a spherically symmetric, static and exact solution of the full nonlinear equations of Einstein without any matter present.

The Schwarzschild solution was immediately accepted as the correct description within general relativity of the gravitational field outside a spherical mass. It would be the correct approximate description of the field around a star such as our sun. But something much more bizzare was implied by the solution. For an object of mass M, the solution appeared to become singular at a radius $R = 2GM/c^2$. For our sun, for example, this radius, now known as the Schwarzschild radius, would be about three kilometers. Now, as long the physical radius of the sun is bigger than three kilometers, the 'Schwarzschild's singularity' is of no concern because inside the sun the Schwarzschild solution is not applicable as there is matter present. But what if the entire mass of the sun was concentrated in a sphere of radius smaller than three kilometers? One would then have to face up to this singularity.

Einstein's reaction to the 'Schwarzschild singularity' was to seek arguments that would make such a singularity inadmissible. Clearly, he believed, a physical theory could not tolerate such singularities. This drove his to write as late as 1939, in a published paper,

"*The essential result of this investigation is a clear understanding as to why the 'Schwarzschild singularities' do not exist in physical reality.*"

This conclusion was however based on an incorrect argument. Einstein was not alone in this rejection of the unpalatable idea of a total gravitational collapse of a physical system. In the same year, in an astronomy conference in Paris, Eddington, one of the leading astronomers of the time, rubbished the work of Chandrasekhar who had concluded from his study of white dwarfs, a work that was to earn him the Nobel prize later, that a large enough star could collapse.

Interestingly, Einstein's paper on the inadmissibility of the Schwarzschild singularity appeared only two months before Oppenheimer and Snyder published their definitive work on stellar collapse with an abstract that read,

"*When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse.*"

Once a sufficiently big star ran out of its nuclear fuel, then there was nothing to stop the inexorable inward pull of gravity. The possibility of stellar collapse meant that a star could be compressed in a region smaller than its Schwarzschild radius and the 'Schwarzschild singularity' could no longer be wished away as Einstein had desired. Indeed it was essential to understand what it means to understand the final state of the star.

It is thus useful to keep in mind what seems now like a mere change of coordinates was at one point a matter of raging intellectual debate.

Chapter 4

Semiclassical Black Holes

In the semiclassical treatment of a black hole, we treat the spacetime geometry of the black hole classically but treat various fields such as the electromagnetic field in this fixed spacetime background quantum mechanically. This semiclassical inclusion of quantum effects already reveals a deep and unexpected connection between the spacetime geometry of a black hole and thermodynamics.

4.1 Hawking temperature

Bekenstein asked a simple-minded but incisive question. If nothing can come out of a black hole, then a black hole will violate the second law of thermodynamics. If we throw a bucket of hot water into a black hole then the net entropy of the world outside would seem to decrease. Do we have to give up the second law of thermodynamics in the presence of black holes?

Note that the energy of the bucket is also lost to the outside world but that does not violate the first law of thermodynamics because the black hole carries mass or equivalently energy. So when the bucket falls in, the mass of the black hole goes up accordingly to conserve energy. This suggests that one can save the second law of thermodynamics if somehow the black hole also has entropy. Following this reasoning and noting the formal analogy between the area of the black hole and entropy discussed in the previous section, Bekenstein proposed that a black hole must have entropy proportional to its area [14].

This way of saving the second law is however in contradiction with the classical properties of a black hole because if a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

For example, for a Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$. Therefore, one would expect inverse temperature that scales as M

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M. \tag{4.1}$$

Now, if the black hole has temperature, then like any hot body, it must radiate. For a classical black hole, by its very nature, this is impossible.

In his seminal work [15], Hawking showed that it *is* possible for a black hole to radiate once quantum effects are included. In quantum theory, particle-antiparticle are constantly being created and annihilated even in vacuum. Near the horizon, an antiparticle can fall in once in a while and the particle can escapes to infinity. In fact, Hawking's calculation showed that the spectrum emitted by the black hole is precisely thermal with temperature $T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}$. With this precise relation between the temperature and surface gravity the laws of black hole mechanics discussed in the earlier section become identical to the laws of thermodynamics. Using the formula for the Hawking temperature and the first law of thermodynamics

$$dM = TdS = \frac{\kappa\hbar}{8\pi G\hbar} dA,$$

one can then deduce the precise relation between entropy and the area of the black hole:

$$S = \frac{Ac^3}{4G\hbar}.$$

Before discussing the entropy of a black hole, let us derive the Hawking temperature in a somewhat heuristic way using a Euclidean continuation of the near horizon geometry. In quantum mechanics, for a system with Hamiltonian H, the thermal partition function is

$$Z = \mathrm{Tr} e^{-\beta \hat{H}}, \tag{4.2}$$

where β is the inverse temperature. This is related to the time evolution operator $e^{-itH/\hbar}$ by a Euclidean analytic continuation $t = -i\tau$ if we identify $\tau = \beta\hbar$. Let us consider a single scalar degree of freedom Φ, then one can write the trace as

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi < \phi | e^{-\tau_E \hat{H}/\hbar} | \phi >$$

and use the usual path integral representation for the propagator to find

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi \int D\Phi e^{-S_E[\Phi]}.$$

Here $S_E[\Phi]$ is the Euclidean action over periodic field configurations that satisfy the boundary condition

$$\Phi(\beta\hbar) = \Phi(0) = \phi.$$

This gives the relation between the periodicity in Euclidean time and the inverse temperature,

$$\beta\hbar = \tau \quad \text{or} \quad T = \frac{\hbar}{\tau}. \tag{4.3}$$

Let us now look at the Euclidean Schwarzschild metric by substituting $t = -it_E$. Near the horizon the line element (3.11) looks like

$$ds^2 = \rho^2 \kappa^2 dt_E^2 + d\rho^2.$$

If we now write $\kappa t_E = \theta$, then this metric is just the flat two-dimensional Euclidean metric written in polar coordinates provided the angular variable θ has the correct periodicity $0 < \theta < 2\pi$. If the periodicity is different, then the geometry would have a conical singularity at $\rho = 0$. This implies that Euclidean time t_E has periodicity $\tau = \frac{2\pi}{\kappa}$. Note that far away from the black hole at asymptotic infinity the Euclidean metric is flat and goes as $ds^2 = d\tau_E^2 + dr^2$. With periodically identified Euclidean time, $t_E \sim t_E + \tau$, it looks like a cylinder. Near the horizon at $\rho = 0$ it is nonsingular and looks like flat space in polar coordinates for this correct periodicity. The full Euclidean geometry thus looks like a cigar. The tip of the cigar is at $\rho = 0$ and the geometry is asymptotically cylindrical far away from the tip.

Using the relation between Euclidean periodicity and temperature, we then conclude that Hawking temperature of the black hole is

$$T = \frac{\hbar\kappa}{2\pi}. \tag{4.4}$$

4.2 Bekenstein-Hawking entropy

Even though we have "derived" the temperature and the entropy in the context of Schwarzschild black hole, this beautiful relation between area and entropy is true quite generally essentially because the near horizon geometry is always Rindler-like. For *all* black holes with charge, spin and in number of dimensions, the Hawking temperature and the entropy are given in terms of the surface gravity and horizon area by the formulae

$$T_H = \frac{\hbar\kappa}{2\pi}, \quad S = \frac{A}{4G\hbar}.$$

This is a remarkable relation between the thermodynamic properties of a black hole on one hand and its geometric properties on the other.

The fundamental significance of entropy stems from the fact that even though it is a quantity defined in terms of gross thermodynamic properties, it contains nontrivial information about the *microscopic* structure of the theory through Boltzmann relation

$$S = k \log(d),$$

where d is the the degeneracy or the total number of microstates of the system of for a given energy, and k is Boltzmann constant. Entropy is not a kinematic quantity like energy or momentum but rather contains information about the total number microscopic degrees of freedom of the system. Because of the Boltzmann relation, one can learn a great deal about the microscopic properties of a system from its thermodynamics properties.

The Bekenstein-Hawking entropy behaves in every other respect like the ordinary thermodynamic entropy. It is therefore natural to ask what microstates might account for it. Since the entropy formula is given by this beautiful, general form

$$S = \frac{Ac^3}{4G\hbar},$$

that involves all three fundamental dimensionful constants of nature, it is a valuable piece of information about the degrees of freedom of a quantum theory of gravity.

4.3 Laws of black hole mechanics

One of the remarkable properties of black holes is that one can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is quite surprising because *a priori* there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

(0) Zeroth Law: In thermal physics, the zeroth law states that the temperature T of a body at thermal equilibrium is constant throughout the

body. Otherwise heat will flow from hot spots to the cold spots. Correspondingly for stationary black holes one can show that surface gravity κ is constant on the event horizon. This is obvious for spherically symmetric horizons but is true also more generally for non-spherical horizons of spinning black holes.

(1) First Law: Energy is conserved, $dE = TdS + \mu dQ + \Omega dJ$, where E is the energy, Q is the charge with chemical potential μ and J is the spin with chemical potential Ω. Correspondingly for black holes, one has $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$. For a Schwarzschild black hole we have $\mu = \Omega = 0$ because there is no charge or spin.

(2) Second Law: In a physical process the total entropy S never decreases, $\Delta S \geq 0$. Correspondingly for black holes one can prove the area theorem that the net area in any process never decreases, $\Delta A \geq 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem, since the area is proportional to the square of the mass, and $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

Thus the laws of black hole mechanics, crystallized by Bardeen, Carter, Hawking, and other bears a striking resemblance with the three laws of thermodynamics for a body in thermal equilibrium. We summarize these results below in Table(4.1) for a black hole of mass M, spin J, and charge Q.

Table 4.1: Classical Laws of Black Hole Mechanics

Laws of Thermodynamics	**Laws of Black Hole Mechanics**
Temperature is constant throughout a body at equilibrium. T= constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ.$	Energy is conserved. $dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ.$
Entropy never decreases. $\Delta S \geq 0.$	Area never decreases. $\Delta A \geq 0.$

Here A is the area of the horizon, and κ is the surface gravity which can be thought of roughly as the acceleration at the horizon, μ is the chemical potential conjugate to Q, and Ω is the angular speed conjugate to J.

We will see that this formal analogy between the laws of black hole mechanics and thermodynamics is actually much more than an analogy. Bekenstein and Hawking discovered that there is a deep connection between black hole geometry, thermodynamics and quantum mechanics. Quantum mechanically, a black hole is not quite black.

4.4 Extremal Black Holes

Reissner-Nordström (RN) black hole

1. *Identify the horizon for this metric and examine the near horizon geometry to show that it has two-dimensional Rindler spacetime as a factor.*
2. *Using the relation to the Rindler geometry determine the surface gravity κ as for the Schwarzschild black hole and thereby determine the temperature and entropy of the black hole.*

$$\begin{aligned} T &= \frac{\kappa\hbar}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{2\pi(2M(M + \sqrt{M^2 - Q^2}) - Q^2)} \\ S &= \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2. \end{aligned}$$

   Recover the formulae for Schwarzschild black hole in the limit $Q = 0$.
3. *Show that in the extremal limit $M \to Q$ the temperature vanishes but the entropy has a nonzero limit. Show that for the extremal Reissner-Nordström black hole the near horizon geometry is of the form $AdS_2 \times S^2$.*

For a physically sensible definition of temperature and entropy in (4.5) the mass must satisfy the bound $M^2 \geq Q^2$. Something special happens when this bound is saturated and $M = |Q|$. In this case $r_+ = r_- = |Q|$ and the two horizons coincide. We choose Q to be positive. The solution (3.21) then takes the form,

$$ds^2 = -(1 - Q/r)^2 dt^2 + \frac{dr^2}{(1 - Q/r)^2} + r^2 d\Omega^2, \tag{4.5}$$

with a horizon at $r = Q$. In this extremal limit (4.5), we see that the temperature of the black hole goes to zero and it stops radiating but nevertheless its entropy has a finite limit given by $S \to \pi Q^2$. When the temperature goes to zero, thermodynamics does not really make sense but we can use this limiting entropy as the definition of the zero temperature entropy.

For extremal black holes it is sometimes more convenient to use isotropic coordinates in which the line element takes the form

$$ds^2 = H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}^2$$

where $d\vec{x}^2$ is the flat Euclidean line element $\delta_{ij}dx^i dx^j$ and $H(\vec{x})$ is a harmonic function of the flat Laplacian

$$\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

The extremal Reissner-Nordström solution is obtained by choosing

$$H(\vec{x}) = \left(1 + \frac{Q}{\rho}\right),$$

and the field strength is given by $F_{0i} = \partial_i H(\vec{x})$.

4.5 Multi-centered Black Holes

One can in fact write a multi-centered Reissner-Nordström solution by choosing a more general harmonic function

$$H = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{x}i|}. \tag{4.6}$$

The total mass M equals the total charge Q and is given additively

$$Q = \sum Q_i. \tag{4.7}$$

The solution is static because the electrostatic repulsion between different centers balances the gravitational attraction between them.

4.6 Anti de Sitter Spacetime and Holography

Note that the coordinate ρ in the isotropic coordinates should not be confused with the coordinate r in the spherical coordinates. In the isotropic coordinates the line-element is

$$ds^2 = -\left(1 + \frac{Q}{\rho}\right)^2 dt^2 + (1 + \frac{Q}{\rho})^{-2}(d\rho^2 + \rho^2 d\Omega^2),$$

and the horizon occurs at $\rho = 0$. Contrast this with the metric in the spherical coordinates (4.5) that has the horizon at $r = Q$. The near horizon geometry is quite different from that of the Schwarzschild black hole. The line element is

$$ds^2 = -\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}(d\rho^2 + \rho^2 d\Omega^2) \tag{4.8}$$

$$= (-\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}dr^2) + (Q^2 d\Omega^2). \tag{4.9}$$

The geometry thus factorizes as for the Schwarzschild solution. One factor the 2-sphere S^2 of radius Q but the other (r, t) factor is now not Rindler any more but is a two-dimensional Anti-de Sitter or AdS_2. The geodesic radial distance in AdS_2 is $\log r$. As a result the geometry looks like an infinite throat near $r = 0$ and the radius of the mouth of the throat has radius Q.

Extremal black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. We now try to see if the entropy can be explained by counting of microstates. In doing so, supersymmetry proves to be a very useful tool.

Chapter 5

Quantum Fields and Topology

5.1 Supersymmetric Quantum Mechanics

5.2 Topological Invariants

5.3 SCFT and Elliptic genus of $K3$

5.4 Combinatorics and Quantum Mechanics

Chapter 6

Elements of String Theory

6.1 BPS states in $\mathcal{N} = 4$ compactifications

Superstring theories are naturally formulated in ten-dimensional Lorentzian spacetime $\mathcal{M}_{10}$. A 'compactification' to four-dimensions is obtained by taking $\mathcal{M}_{10}$ to be a product manifold $\mathbb{R}^{1,3} \times X_6$ where X_6 is a compact Calabi-Yau threefold and $\mathbb{R}^{1,3}$ is the noncompact Minkowski spacetime. We will focus in these lectures on a compactification of Type-II superstring theory when X_6 is itself the product $X_6 = K3 \times T^2$. A highly nontrivial and surprising result from the 90s is the statement that this compactification is quantum equivalent or 'dual' to a compactification of heterotic string theory on $T^4 \times T^2$ where T^4 is a four-dimensional torus [21, 22]. One can thus describe the theory either in the Type-II frame or the heterotic frame.

The four-dimensional theory in $\mathbb{R}^{1,3}$ resulting from this compactification has $\mathcal{N} = 4$ supersymmetry[1]. The massless fields in the theory consist of 22 vector multiplets in addition to the supergravity multiplet. The massless moduli fields consist of the S-modulus λ taking values in the coset

$$SL(2, \mathbb{Z}) \backslash SL(2; \mathbb{R})/O(2; \mathbb{R}), \tag{6.1}$$

and the T-moduli μ taking values in the coset

$$O(22, 6; \mathbb{Z}) \backslash O(22, 6; \mathbb{R})/O(22; \mathbb{R}) \times O(6; \mathbb{R}). \tag{6.2}$$

[1] This supersymmetry is a super Lie algebra containing $ISO(1,3) \times SU(4)$ as the bosonic subalgebra where $ISO(1,3)$ is the Poincaré symmetry of the $\mathbb{R}^{1,3}$ spacetime and $SU(4)$ is an internal symmetry called R-symmetry in physics literature. The odd generators of the superalgebra are called supercharges. With $\mathcal{N} = 4$ supersymmetry, there are eight complex supercharges which transform as a spinor of $ISO(1,3)$ and a fundamental of $SU(4)$.

The group of discrete identifications $SL(2, \mathbb{Z})$ is called S-duality group. In the heterotic frame, it is the electro-magnetic duality group [23, 24] whereas in the type-II frame, it is simply the group of area- preserving global diffeomorphisms of the T^2 factor. The group of discrete identifications $O(22, 6; \mathbb{Z})$ is called the T-duality group. Part of the T-duality group $O(19, 3; \mathbb{Z})$ can be recognized as the group of geometric identifications on the moduli space of K3; the other elements are stringy in origin and have to do with mirror symmetry.

At each point in the moduli space of the internal manifold $K3 \times T^2$, one has a distinct four- dimensional theory. One would like to know the spectrum of particle states in this theory. Particle states are unitary irreducible representations, or supermultiplets, of the $\mathcal{N} = 4$ superalgebra. The supermultiplets are of three types which have different dimensions in the rest frame. A long multiplet is 256- dimensional, an intermediate multiplet is 64-dimensional, and a short multiplet is 16- dimensional. A short multiplet preserves half of the eight supersymmetries (*i.e.* it is annihilated by four supercharges) and is called a half-BPS state; an intermediate multiplet preserves one quarter of the supersymmetry (*i.e.* it is annihilated by two supercharges), and is called a quarter-BPS state; and a long multiplet does not preserve any supersymmetry and is called a non-BPS state. One consequence of the BPS property is that the spectrum of these states is 'topological' in that it does not change as the moduli are varied, except for jumps at certain walls in the moduli space [25].

An important property of the BPS states that follows from the superalgebra is that their mass is determined by the charges and the moduli [25]. Thus, to specify a BPS state at a given point in the moduli space, it suffices to specify its charges. The charge vector in this theory transforms in the vector representation of the T-duality group $O(22, 6; \mathbb{Z})$ and in the fundamental representation of the S-duality group $SL(2, \mathbb{Z})$. It is thus given by a vector $\Gamma^{i\alpha}$ with integer entries

$$\Gamma^{i\alpha} = \begin{pmatrix} Q^i \\ P^i \end{pmatrix} \quad \textit{where} \quad i = 1, 2, \ldots 28; \quad \alpha = 1, 2 \tag{6.3}$$

transforming in the $(2, 28)$ representation of $SL(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z})$. The vectors Q and P can be regarded as the quantized electric and magnetic charge vectors of the state respectively. They both belong to an even, integral, self-dual lattice $\Pi^{22,6}$. We will assume in what follows that $\Gamma = (Q, P)$ in (6.3) is primitive in that it cannot be written as an integer multiple of (Q_0, P_0) for Q_0 and P_0 belonging to $\Pi^{22,6}$. A state is called purely electric if only Q is non-zero, purely magnetic if only P is non- zero, and dyonic if both P and Q are non-zero.

To define S-duality transformations, it is convenient to represent the S-modulus as a complex field S taking values in the upper half plane. An S-duality transformation

$$\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}) \tag{6.4}$$

acts simultaneously on the charges and the S-modulus by

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}; \qquad S \to \frac{aS+b}{cS+d} \tag{6.5}$$

To define T-duality transformations, it is convenient to represent the T-moduli by a 28×28 of matrix μ_I^A satisfying

$$\mu^t \, L \, \mu = L \tag{6.6}$$

with the identification that $\mu \sim k\mu$ for every $k \in O(22;\mathbb{R}) \times O(6;\mathbb{R})$. Here L is the (28×28) matrix

$$L_{IJ} = \begin{pmatrix} -\mathbf{C}_{16} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_6 \\ \mathbf{0} & \mathbf{I}_6 & \mathbf{0} \end{pmatrix}, \tag{6.7}$$

with $\mathbf{I}_s$ the $s \times s$ identity matrix and $\mathbf{C}_{16}$ is the Cartan matrix of $E_8 \times E_8$. The T-moduli are then represented by the matrix

$$\mathcal{M} = \mu^t \mu \tag{6.8}$$

which satisifies

$$\mathcal{M}^t = \mathcal{M}, \qquad \mathcal{M}^t L \mathcal{M} = L \tag{6.9}$$

In this basis, a T-duality transformation can then be represented by a (28×28) matrix R with integer entries satisfying

$$R^t L R = L, \tag{6.10}$$

which acts simultaneously on the charges and the T-moduli by

$$Q \to RQ; \quad P \to RP; \quad \mu \to \mu R^{-1} \tag{6.11}$$

Given the matrix μ_I^A, one obtains an embedding $\Lambda^{22,6} \subset \mathbb{R}^{22,6}$ of $\Pi^{22,6}$ which allows us to define the moduli-dependent charge vectors Q and P by

$$Q^A = \mu_I^A Q_I \quad P^A = \mu_I^A P_I \, . \tag{6.12}$$

Note that while Q^I are integers Q^A are not. In what follows we will not always write the indices explicitly assuming that it will be clear from the context. In any case, the final answers will only depend on the T-duality invariants which are all integers. The matrix L has a 22-dimensional eigensubspace with eigenvalue -1 and a 6- dimensional eigensubspace with eigenvalue $+1$. Given Q and P, one can define the 'right-moving' charges[2] Q_R and P_R as the projections of Q and P respectively onto the subspace with eigenvalue $+1$. and the 'left-moving' charges as projections onto the subspace with eignevalue -1. These definitions can be compactly written as

$$Q_{R,L} = \frac{(1 \pm L)}{2} Q \, ; \quad P_{R,L} = \frac{(1 \pm L)}{2} P \tag{6.13}$$

The right-moving charges since for the heterotic string, Q_R are related to the right-moving momenta. The central charges Z_1 and Z_2 of the $\mathcal{N} = 4$ superalgebra can then be defined in terms of the right-moving charges and moduli (For details of these definitions and the superalgebra, see §8.1).

If the vectors Q and P are nonparallel, then the state is quarter-BPS. On the other hand, if $Q = pQ_0$ and $P = qQ_0$ for some $Q_0 \in \Pi^{22,6}$ with p and q relatively prime integers, then the state is half-BPS.

An important piece of nonperturbative information about the dynamics of the theory is the exact spectrum of all possible dyonic BPS- states at all points in the moduli space. More specifically, one would like to compute the number $d(\Gamma)|_{\lambda,\mu}$ of dyons of a given charge Γ at a specific point (λ, μ) in the moduli space. Computation of these numbers is of course a very complicated dynamical problem. In fact, for a string compactification on a general Calabi-Yau threefold, the answer is not known. One main reason for focusing on this particular compactification on $K3 \times T^2$ is that in this case the dynamical problem has been essentially solved and the exact spectrum of dyons is now known. Furthermore, the results are easy to summarize and the numbers $d(\Gamma)|_{\lambda,\mu}$ are given in terms of Fourier coefficients of various modular forms.

In view of the duality symmetries, it is useful to classify the inequivalent duality orbits labeled by various duality invariants. This leads to an interesting problem in number theory of classification of inequivalent duality orbits of various duality groups such as $SL(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z})$ in our case and more exotic groups like $E_{7,7}(\mathbb{Z})$ for other choices of compactification manifold X_6. It is important to remember though that a duality transformation acts simultaneously on charges and the moduli. Thus, it maps a state with charge Γ at a point in the moduli space (λ, μ) to a state with charge Γ' but at some

[2] The right- moving charges couple to the graviphoton vector fields associated with the right-moving chiral currents in the conformal field theory of the dual heterotic string.

other point in the moduli space (λ', μ'). In this respect, the half-BPS and quarter-BPS dyons behave differently.

- For half-BPS states, the spectrum does not depend on the moduli. Hence $d(\Gamma)|_{\lambda',\mu'} = d(\Gamma)|_{\lambda,\mu}$. Furthermore, by an S-duality transformation one can choose a frame where the charges are purely electric with $P = 0$ and $Q \neq 0$. Single-particle states have Q primitive and the number of states depends only on the T-duality invariant integer $n \equiv Q^2/2$. We can thus denote the degeneracy of half-BPS states $d(\Gamma)|_{S',\mu'}$ simply by $d(n)$.

- For quarter-BPS states, the spectrum does depend on the moduli, and $d(\Gamma)|_{\lambda',\mu'} \neq d(\Gamma)|_{\lambda,\mu}$. However, the partition function turns out to be independent of moduli and hence it is enough to classify the inequivalent duality orbits to label the partition functions. For the specific duality group $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$ the partition functions are essentially labeled by a single discrete invariant [26, 27, 28].

$$I = \gcd(Q \wedge P)\,, \tag{6.14}$$

The degeneracies themselves are Fourier coefficients of the partition function. For a given value of I, they depend only on[3] the moduli and the three T-duality invariants $(m, n, \ell) \equiv (P^2/2, Q^2/2, Q \cdot P)$. Integrality of (m, n, ℓ) follows from the fact that both Q and P belong to $\Pi^{22,6}$. We can thus denote the degeneracy of these quarter-BPS states $d(\Gamma)|_{\lambda,\mu}$ simply by $d(m, n, l)|_{\lambda,\mu}$. For simplicity, we consider only $I = 1$ in these lectures. Generalization for higher I can be found in [29, 30].

6.2 Exercises

Elements of string compactifications

The heterotic string theory in ten dimensions has 16 supersymmetries. The bosonic massless fields consist of the metric g_{MN}, a 2-form field $B^{(2)}$, 16 abelian 1-form gauge fields $A^{(r)}$ $r = 1, \ldots 16$, and a real scalar field ϕ called the dilaton. The Type-IIB string theory in ten dimensions has 32 supersymmetries. The bosonic massless fields consist of the metric g_{MN}; two 2-form fields $C^{(2)}, B^{(2)}$; a self-dual 4-form field $C^{(4)}$; and a complex scalar field λ called the dilaton-axion field.

[3] There is an additional dependence on arithmetic T-duality invariants but the degeneracies for states with nontrivial values of these T-duality invariants can be obtained from the degeneracies discussed here by demanding S-duality invariance [28].

One of the remarkable strong-weak coupling dualities is the 'string-string' duality between heterotic string compactified on $T^4 \times T^2$ and Type-IIB string compactified on $K3 \times T^2$. One piece of evidence for this duality is obtained by comparing the massless spectrum for these compactifications and certain half-BPS states in the spectrum.

1. *Show that the heterotic string compactified on* $T^4 \times S^1 \times \tilde{S}^1$ *leads a four dimensional theory with* $\mathcal{N} = 4$ *supersymmetry with* 22 *vector multiplets.*
2. *Show that the Type-IIB string compactified on* $K3 \times S^1 \times \tilde{S}^1$ *leads a four dimensional theory with* $\mathcal{N} = 4$ *supersymmetry with* 22 *vector multiplets.*
3. *Show that the Kaluza-Klein monopole in Type-IIB string associated with the circle* $\tilde{S}^1$ *has the right structure of massless fluctuations to be identified with the half-BPS perturbative heterotic string in the dual description.*

6.3 String-String duality

It will be useful to recall a few details of the string-string duality between heterotic compactified on $T^4 \times S^1 \times \tilde{S}^1$ and Type-IIB compactified on $K3 \times S^1 \times \tilde{S}^1$. Two pieces of evidence for this duality will be relevant to our discussion.

• *Low energy effective action*

Both these compactifications result in $\mathcal{N} = 4$ supergravity in four dimensions. With this supersymmetry, the two-derivative effective action for the massless fields receives no quantum corrections. Hence, if the two theories are to be dual to each other, they must have identical 2-derivative action.

This is indeed true. Even though the field content and the action are very different for the two theories in ten spacetime dimensions, upon respective compactifications, one obtains $\mathcal{N} = 4$ supergravity with 22 vector multiplets coupled to the supergravity multiplet. This has been discussed briefly in one of the tutorials. For a given number of vector multiplets, the two-derivative action is then completely fixed by supersymmetry and hence is the same for the two theories. This was one of the properties that led to the conjecture of a strong-weak coupling duality between the two theories.

For our purposes, we will be interested in the 2-derivative action for the bosonic fields. This is a generalization of the Einstein-Hilbert-Maxwell action

(3.1) which couples the metric, the moduli fields and 28 abelian gauge fields:

$$\begin{aligned} I &= \frac{1}{32\pi}\int d^4x\sqrt{-\det G}\, S\,[R_G + \frac{1}{S^2}G^{\mu\nu}(\partial_\mu S\partial_\nu S - \frac{1}{2}\partial_\mu a\partial_\nu a) \\ &\quad + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML) - G^{\mu\mu'}G^{\nu\nu'}F^{(i)}_{\mu\nu}(LML)_{ij}F^{(j)}_{\mu'\nu'} \\ &\quad - \frac{a}{S}G^{\mu\mu'}G^{\nu\nu'}F^{(i)}_{\mu\nu}L_{ij}\tilde{F}^{(j)}_{\mu'\nu'}] \qquad i,j = 1,\ldots,28. \end{aligned} \tag{6.15}$$

In the heterotic string picture, the expectation value of the dilaton field S is related to the four-dimensional string coupling g_4

$$S \sim \frac{1}{g_4^2}\,, \tag{6.16}$$

and a is the axion field. The metric $G_{\mu\nu}$ is the metric in the string frame and is related to the metric $g_{\mu\nu}$ in Einstein frame by the Weyl rescaling

$$g_{\mu\nu} = SG_{\mu\nu} \tag{6.17}$$

- *BPS spectrum*

Another requirement of duality is that the spectrum of BPS states should match for the two dual theories. Perturbative states in one description will generically get mapped to some non-perturbative states in the dual description. As a result, this leads to highly nontrivial predictions about the nonpertubative spectrum in the dual description given the perturbative spectrum in one description.

As an example, consider the perturbative BPS-states in heterotic string theory on $K3 \times S^1 \times \tilde{S}^1$. A heterotic string wrapping w times on S^1 and carrying momentum n gets mapped in Type-IIA to the NS5-brane wrapping w times on $K3\times S^1$ and carrying momentum n. One can go from Type-IIA to Type-IIB by a T-duality along the $\tilde{S}^1$ circle. Under this T-duality, the NS5-brane gets mapped to a KK-monopole with monopole charge w associated with the circle $\tilde{S}^1$ and carrying momentum n. This thus leads to a prediction that the spectrum of KK-monopole carrying momentum in Type-IIB should be the same as the spectrum of perturbative heterotic string discussed earlier. We will verify this highly nontrivial prediction in the next subsection for the case of $w = 1$.

6.4 Supersymmetry and extremality

Some of the special properties of external black holes can be understood better by embedding them in supergravity. We will be interested in these lectures in string compactifications with $\mathcal{N} = 4$ supersymmetry in four spacetime dimensions. The $\mathcal{N} = 4$ supersymmetry algebra contains in addition to the usual Poincaré generators, sixteen real supercharges which can be grouped into 8 complex charges Q^a_α and their complex conjugates. Here $\alpha = 1, 2$ is the usual Weyl spinor index of 4d Lorentz symmetry. and the internal index $a = 1, \ldots, 4$ in the fundamental **4** representation of an $SU(4)$, the R-symmetry of the superalgebra. The relevant anticommutators for our purpose are

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2P_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta^a_b$$

$$\{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta} Z^{ab} \qquad \{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = \bar{Z}_{ab}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{6.18}$$

where σ^μ are (2×2) matrices with $\sigma_0 = -\mathbf{1}$ and $\sigma^i fori = 1, 2, 3$ are the usual Pauli matrices. Here P_μ is the momentum operator and Q are the supersymmetry generators and the complex number Z^{ab} is the central charge matrix.

Let us first look at the representations of this algebra when the central charge is zero. In this case the massive and massless representation are qualitatively different.

1. Massive Representation, $M > 0, P^\mu = (M, 0, 0, 0)$
   In this case, (6.18) becomes $\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = 2M\delta_{\alpha\dot{\beta}}\delta^a_b$ and all other anticommutators vanish. Up to overall scaling, these are the commutation relations for eight complex fermionic oscillators. Each oscillator has a two-state representation, which is either filled or empty. These states together define a unitary irreducible representation, called a supermultiplet, of the superalgebra. The total dimension of the representation is $2^8 = 256$ which is CPT self-conjugate.

2. Massless Representation $M = 0, P^\mu = (E, 0, 0, E)$
   In this case (6.18) becomes $\{Q^a_1, \bar{Q}_{\dot{1}b}\} = 2E\delta^a_b$ and all other anticommutators vanish. Up to overall scaling, these are now the anticommutation relations of *four* fermionic oscillators and hence the total dimension of the representation is $2^4 = 16$ which is also CPT-self-conjugate.

The important point is that for a massive representation, with $M = \epsilon > 0$, no matter how small ϵ, the supermultiplet is long and precisely at $M = 0$ it is short. Thus the size of the supermultiplet has to change discontinuously if the state has to acquire mass. Furthermore, the size of the supermultiplet is determined by the number of supersymmetries that are *broken* because those have non-vanishing anti-commutations and turn into fermionic oscillators.

Note that there is a bound on the mass $M \geq 0$ which simply follows from the fact the using (6.18) one can show that the mass operator on the right hand side of the equation equals a positive operator, the absolute value square of the supercharge on the left hand side. The massless representation saturates this bound and is 'small' whereas the massive representation is long.

There is an analog of this phenomenon also for nonzero Z_{ab}. As explained in the appendix, the central charge matrix Z_{ab} can be brought to the standard form by an $U(4)$ rotation

(6.19)

$$\tilde{Z} = UZU^T, \;\; U \in U(4)\;, \qquad \tilde{Z}_{ab} = \left(\begin{array}{c|c} Z_1\varepsilon & 0 \\ \hline 0 & Z_2\varepsilon \end{array}\right)\;, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\;.$$

so we have two 'central charges' Z_1 and Z_2. Without loss of generality we can assume $|Z_1| \geq |Z_2|$. Using the supersymmetry algebra one can prove the BPS bound $M - |Z_1| \geq 0$ by showing that this operator is equal to a positive operator (see appendix for details). States that saturate this bound are the BPS states. There are three types of representations:

- If $M = |Z_1| = |Z_2|$, then eight of of the sixteen supersymmetries are preserved. Such states are called half-BPS. The broken supersymmetries result in four complex fermionic zero modes whose quantization furnishes a 2^4-dimensional short multiplet

- If $M = |Z_1| > |Z_2|$, then and four out of the sixteen supersymmetries are preserved. Such states are called quarter-BPS. The broken supersymmetries result in six complex fermionic zero modes whose quantization furnishes a 2^6-dimensional intermediate multiplet.

- If $M > |Z_1| > |Z_2|$, then no supersymmetries are preserved. Such states are called non-BPS.The sixteen broken supersymmetries result in eight complex fermionic zero modes whose quantization furnishes a 2^8-dimensional long multiplet.

The significance of BPS states in string theory and in gauge theory stems from the classic argument of Witten and Olive which shows that under suitable conditions, the spectrum of BPS states is stable under smooth changes of moduli and coupling constants. The crux of the argument is that with sufficient supersymmetry, for example $\mathcal{N} = 4$, the coupling constant does not get renormalized. The central charges Z_1 and Z_2 of the supersymmetry algebra depend on the quantized charges and the coupling constant which therefore also does not get renormalized. This shows that for BPS states, the mass also cannot get renormalized because if the quantum corrections increase the mass, the states will have to belong a long representation . Then, the number of states will have to jump discontinuously from, say from 16 to 256 which cannot happen under smooth variations of couplings unless there is some kind of a 'Higgs Mechanism' or there is some kind of a phase transition[4]

As a result, one can compute the spectrum at weak coupling in the region of moduli space where perturbative or semiclassical counting methods are available. One can then analytically continue this spectrum to strong coupling. This allows us to obtain invaluable non-perturbative information about the theory from essentially perturbative commutations.

6.5 BPS dyons in $\mathcal{N} = 4$ compactifications

The massless spectrum of the toroidally compactified heterotic string on T^6 contains 28 different "photons" or $U(1)$ gauge fields – one from each of the 22 vector multiplets and 6 from the supergravity multiplet. As a result, the electric charge of a state is specified by a 28-dimensional charge vector Q and the magnetic charge is specified by a 28-dimensional charge vector P. Thus, a dyonic state is specified by the charge vector

$$\Gamma = \begin{pmatrix} Q \\ P \end{pmatrix} \tag{6.20}$$

where Q and P are the electric and magnetic charge vectors respectively. Both Q and P are elements of a self-dual integral lattice $\Pi^{22,6}$ and can be represented as 28-dimensional column vectors in $\mathbb{R}^{22,6}$ with integer entries, which transform in the fundamental representation of $O(22, 6; \mathbb{Z})$. We will be interested in BPS states.

- For half-BPS state the charge vectors Q and P must be parallel. These states are dual to perturbative BPS states.
- For a quarter-BPS states the charge vectors Q and P are not parallel. There is no duality frame in which these states are perturbative.

[4]Such 'phase transitions' do occur and the degeneracies can jump upon crossing certain walls in the moduli space. This phenomenon called 'wall-crossing' occurs not because of Higgs mechanism but because at the walls, single particle states have the same mass as certain multi-particle states and can thus mix with the multi-particle continuum states. The wall-crossing phenomenon complicates the analytic continuation of the degeneracy from weak coupling from strong coupling since one may encounter various walls along the way. However, in many cases, the jumps across these walls can be taken into account systematically.

There are three invariants of $O(22,6;\mathbb{Z})$, quadratic in charges, and given by P^2, Q^2 and $Q \cdot P$. These three T-duality invariants will be useful in later discussions.

Chapter 7

Spectrum of Half-BPS Dyons

An instructive example of BPS of states is provided by an infinite tower of BPS states that exists in perturbative string theory [31, 32].

7.1 Perturbative half-BPS States

Consider a perturbative heterotic string state wrapping around S^1 with winding number w and quantized momentum n. Let the radius of the circle be R and $\alpha' = 1$, then one can define left-moving and right-moving momenta as usual,

$$p_{L,R} = \sqrt{\frac{1}{2}}\left(\frac{n}{R} \pm wR\right). \tag{7.1}$$

Recall that the heterotic strings consists of a right-moving superstring and a left-moving bosonic string. In the NSR formalism in the light-cone gauge, the worldsheet fields are:

- Right moving superstring $X^i(\sigma^-)$ $\tilde{\psi}^i(\sigma^-)$ $\qquad i = 1 \cdots 8$
- Left-moving bosonic string $X^i(\sigma^+), X^I(\sigma^+)$ $\qquad I = 1 \cdots 16,$

where X^i are the bosonic transverse spatial coordinates, $\tilde{\psi}^i$ are the worldsheet fermions, and X^I are the coordinates of an internal $E_8 \times E_8$ torus. A BPS state is obtained by keeping the right-movers in the ground state (that is, setting the right-moving oscillator number $\tilde{N} = \frac{1}{2}$ in the NS sector and $\tilde{N} = 0$ in the R sector).

The Virasoro constraints are then given by

$$\tilde{L}_0 - \frac{M^2}{4} + \frac{p_R^2}{2} = 0 \tag{7.2}$$

$$L_0 - \frac{M^2}{4} + \frac{p_L^2}{2} = 0, \tag{7.3}$$

where N and $\tilde{N}$ are the left-moving and right-moving oscillation numbers respectively.

The left-moving oscillator number is then

$$L_0 = \sum_{n=1}^{\infty}(\sum_{i=1}^{8} n a_{-n}^i a_n^i + \sum_{I=1}^{16} n\beta_{-n}^I \beta_{-n}^I) - 1 := N - 1, \tag{7.4}$$

where a^i are the left-moving Fourier modes of the fields X^i, and β^I are the Fourier modes of the fields X^I. Note that the right-moving fermions satisfy anti-periodic boundary condition in the NS sector and have half-integral moding, and satisfy periodic boundary conditions in the R sector and have integral moding. The oscillator number operator is then given by

$$\tilde{L}_0 = \sum_{n=1}^{\infty}\sum_{i=1}^{8}(n\tilde{a}_{-n}^i \tilde{a}_n^i + r\tilde{\psi}_{-r}^i \tilde{\psi}_r^i - \frac{1}{2}) := \tilde{N} - \frac{1}{2}. \tag{7.5}$$

with $r \equiv -(n - \frac{1}{2})$ in the NS sector and by

$$\tilde{L}_0 = \sum_{n=1}^{\infty}\sum_{i=1}^{8}(n\tilde{a}_{-n}^i \tilde{a}_n^i + r\tilde{\psi}_{-r}^i \tilde{\psi}_r^i) \tag{7.6}$$

with $r \equiv (n - 1)$ in the R sector.

In the NS-sector then one then has $\tilde{N} = \frac{1}{2}$ and the states are given by

$$\tilde{\psi}_{-\frac{1}{2}}^i |0>, \tag{7.7}$$

that transform as the vector representation 8_v of $SO(8)$. In the R sector the ground state is furnished by the representation of fermionic zero mode algebra $\{\psi_0^i, \psi_0^j\} = \delta^{ij}$ which after GSO projection transforms as 8_s of $SO(8)$. Altogether the right-moving ground state is thus 16-dimensional $8_v \oplus 8_s$. From the Virasoro constraint (7.2) we see that a BPS state with $\tilde{N} = 0$ saturates the BPS bound

$$M = \sqrt{2} p_R, \tag{7.8}$$

and thus $\sqrt{2}p_R$ can be identified with the central charge of the supersymmetry algebra. The right-moving ground state after the usual GSO projection is indeed 16-dimensional as expected for a BPS-state in a theory with $\mathcal{N} = 4$ supersymmetry.

We thus have a perturbative BPS state which looks pointlike in four dimensions with two integral charges n and w that couple to two gauge fields $g_{5\mu}$ and $B_{5\mu}$ respectively. It saturates a BPS bound $M = \sqrt{2}p_R$ and belongs to a 16-dimensional short representation. This point-like state is our 'would-be' black hole. Because it has a large mass, as we increase the string coupling it would begin to gravitate and eventually collapse to form a black hole.

Microscopically, there is a huge multiplicity of such states which arises from the fact that even though the right-movers are in the ground state, the string can carry arbitrary left-moving oscillations subject to the Virasoro constraint. Using $M = \sqrt{2}p_R$ in the Virasoro constraint for the left-movers gives us

$$N - 1 = \frac{1}{2}(p_R^2 - p_L^2) := Q^2/2 = nw. \tag{7.9}$$

We would like to know the degeneracy of states for a given value of charges n and w which is given by exciting arbitrary left-moving oscillations whose total worldsheet oscillator excitation number adds up to N. Let us take $w = 1$ for simplicity and denote the degeneracy by $d(n)$ which we want to compute. As usual, it is more convenient to evaluate the canonical partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta L_0}\right) \tag{7.10}$$

$$\equiv \sum_{-1}^{\infty} d(n)q^n \quad q := e^{-\beta}\,. \tag{7.11}$$

This is the canonical partition function of 24 left-moving massless bosons in $1+1$ dimensions at temperature $1/\beta$. The micro-canonical degeneracy $d(N)$ is given then given as usual by the inverse Laplace transform

$$d(N) = \frac{1}{2\pi i}\int d\beta e^{\beta N} Z(\beta). \tag{7.12}$$

Using the expression (7.4) for the oscillator number s and the fact that

$$\mathrm{Tr}(q^{-s\alpha_{-n}\alpha_n}) = 1 + q^s + q^{2s} + q^{3s} + \cdots = \frac{1}{(1-q^s)}\,, \tag{7.13}$$

the partition function can be readily evaluated to obtain

$$Z(\beta) = \frac{1}{q}\prod_{s=1}^{\infty}\frac{1}{(1-q^s)^{24}} \quad . \tag{7.14}$$

It is convenient to introduce a variable τ by $\beta := -2\pi i\tau$, so that$q := e^{2\pi i\tau}$. The function

$$\Delta(\tau) = q\prod_{s=1}^{\infty}(1-q^s)^{24}, \tag{7.15}$$

is the famous discriminant function. Under modular transformations

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad a,b,c,d \in \mathbb{Z}\,, \quad with \quad ad-bc=1 \tag{7.16}$$

it transforms as a modular form of weight 12:

$$\Delta(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{12}\Delta(\tau)\,, \tag{7.17}$$

This remarkable property allows us to relate high temperature ($\beta \to 0$) to low tempreature ($\beta \to \infty$) and derive a simple explicit expression for the asymptotic degeneracies $d(n)$ for n very large.

7.2 Cardy formula

The degeneracy $d(N)$ can be obtained from the canonical partition function by the inverse Laplace transform

$$d(N) = \frac{1}{2\pi i}\int d\beta e^{\beta N} Z(\beta). \tag{7.18}$$

We would like to evaluate this integral (7.18) for large N which corresponds to large worldsheet energy. Such an asymptotic expansion of $d(N)$ for large N is given by the 'Cardy formula' which utilizes the modular properties of the partition function.

For large N, we expect that the integral receives most of its contributions from high temperature or small β region of the integrand. To compute the large N asymptotics, we then need to know the small β asymptotics of the partition function. Now, $\beta \to 0$ corresponds to $q \to 1$ and in this limit the asymptotics of $Z(\beta)$ are very difficult to read off from (7.14) because its a product of many quantities that are becoming very large. It is more convenient to use the fact that $Z(\beta)$ is the inverse of $\Delta(\tau)$ which is a modular form of weight 12 we can conclude

$$Z(\beta) = (\beta/2\pi)^{12} Z(\frac{4\pi^2}{\beta}). \tag{7.19}$$

This allows us to relate the $q \to 1$ or high temperature asymptotics to $q \to 0$ or low temperature asymptotics as follows. Now, $Z(\tilde{\beta}) = Z\left(\frac{4\pi^2}{\beta}\right)$ asymptotics are easy to read off because as $\beta \to 0$ we have $\tilde{\beta} \to \infty$ or $e^{-\tilde{\beta}} = \tilde{q} \to 0$. As $\tilde{q} \to 0$

$$Z(\tilde{\beta}) = \frac{1}{\tilde{q}} \prod_{n=1}^{\infty} \frac{1}{(1-\tilde{q}^n)^{24}} \sim \frac{1}{\tilde{q}}. \tag{7.20}$$

This allows us to write

$$d(N) \sim \frac{1}{2\pi i} \int \left(\frac{\beta}{2\pi}\right)^{12} e^{\beta N + \frac{4\pi^2}{\beta}} d\beta. \tag{7.21}$$

This integral can be evaluated easily using saddle point approximation. The function in the exponent is $f(\beta) \equiv \beta N + \frac{4\pi^2}{\beta}$ which has a maximum at

$$f'(\beta) = 0 \quad \text{or} \quad N - \frac{4\pi^2}{\beta_c} = 0 \quad \text{or} \quad \beta_c = \frac{2\pi}{\sqrt{N}}. \tag{7.22}$$

The value of the integrand at the saddle point gives us the leading asymptotic expression for the number of states

$$d(N) \sim \exp{(4\pi\sqrt{N})}. \tag{7.23}$$

This implies that the ensemble of such BPS states of a given charge vector Q has nonzero statistical entropy that goes to leading order as

$$S_{stat}(Q) := \log(d(Q)) = 4\pi\sqrt{Q^2/2}. \tag{7.24}$$

We would now like to identify the black hole solution corresponding to this state and test if this microscopic entropy agrees with the macroscopic entropy of the black hole.

Chapter 8

Mathematical Background

8.1 $\mathcal{N} = 4$ supersymmetry

We summarize here some facts about the representation of the $\mathcal{N} = 4$ superalgebra. For more details see for example [60].

Massless supermultiplets

There are two massless representations that will be of interest to us.

1. Supergravity multiplet:
   It contains the metric $g_{\mu\nu}$, six vectors $A_\mu^{(ab)}$, and two gravitini $\psi^a_{\mu\alpha}$.

2. Vector Multiplet:
   It contains a vector A_μ, six scalar fields $X^{(ab)}$, and the gaugini χ^a_α,

The low energy massless spectrum of a supergravity theory consists of the supergravity multiplet and n_v vector multiplets. Supersymmetry then completely fixes the form of the two derivative action. The compactification of heterotic string theory on T^6 leads to a theory in four spacetime dimensions with $\mathcal{N} = 4$ supersymmetry and 28 abelian gauge fields which corresponds to $28 - 6 = 22$ vector multiplets.

General BPS representations

In the rest frame of the dyon, the $\mathcal{N} = 4$ supersymmetry algebra takes the form
(8.1)

$$\{Q^a_\alpha, Q^{\dagger b}_{\dot\beta}\} = M\delta_{\alpha\dot\beta}\,\delta^{ab}\ , \quad \{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}Z^{ab}\ , \quad \{Q^{\dagger a}_{\dot\alpha}, Q^{\dagger b}_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\bar{Z}^{ab}$$

where $a, b = 1, \dots 4$ are $SU(4)$ R-symmetry indices and α, β are Weyl spinor indices. In a given charge sector, the central charge matrix encodes information about the charges and the moduli. To write it explicitly, we first define a central charge vector in $\mathcal{C}^6$

$$Z^m(\Gamma) = \frac{1}{\sqrt{\tau_2}}(Q_R^m - \tau P_R^m) \;, \quad m = 1, \dots 6 \;, \tag{8.2}$$

which transforms in the (complex) vector representation of $Spin(6)$. Using the equivalence $Spin(6) = SU(4)$, we can relate it to the antisymmetric representation of Z_{ab} by

$$Z_{ab}(\Gamma) = \frac{1}{\sqrt{\tau_2}}(Q_R - \tau P_R)^m \lambda^m_{ab} \;, \quad m = 1, \dots 6 \tag{8.3}$$

where λ^m_{ab} are the Clebsch-Gordon matrices. Since $Z(\Gamma)$ is antisymmetric, it can be brought to a block-diagonal form by a $U(4)$ rotation

$$\tilde{Z} = UZU^T, \;\; U \in U(4) \;, \qquad \tilde{Z}_{ab} = \left(\begin{array}{c|c} Z_1 \varepsilon & 0 \\ \hline 0 & Z_2 \varepsilon \end{array} \right) \;, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.4}$$

where Z_1 and Z_2 are non-negative real numbers. A $U(2)$ rotation in the 12 plane and another $U(2)$ rotation in the 34 plane will not change the block diagonal form. Since ε is the invariant tensor of $SU(2)$, the $U(2) \times U(2)$ transformation can only change independently the phases of Z_1 and Z_2. We will therefore treat more generally Z_1 and Z_2 as complex numbers.

We now split the $SU(4)$ index as $a = (r, i)$, where $r, i = 1, 2$ and i represents the block number. Defining the following fermionic oscillators

$$\mathcal{A}^i_\alpha = \frac{1}{\sqrt{2}}(\mathcal{Q}^{1i}_\alpha + \epsilon_{\alpha\beta} \mathcal{Q}^{\dagger\, 2i}_\beta), \quad \mathcal{B}^i_\alpha = \frac{1}{\sqrt{2}}(\mathcal{Q}^{1i}_\alpha - \epsilon_{\alpha\beta} \mathcal{Q}^{\dagger\, 2i}_\beta) \;, \quad \mathcal{Q}^a = U^a_b\, Q^b \tag{8.5}$$

the supersymmetry algebra takes the form

$$\{\mathcal{A}^{i\dagger}_{\dot\alpha}, \mathcal{A}^j_\beta\} = (M + Z_i)\, \delta_{\dot\alpha\beta}\, \delta^{ij} \;, \quad \{\mathcal{B}^{i\dagger}_{\dot\alpha}, \mathcal{B}^j_\beta\} = (M - Z_i)\, \delta_{\dot\alpha\beta}\, \delta^{ij} \tag{8.6}$$

with all other anti-commutators being zero.

Let us conclude by giving an explicit representation for λ^m_{ab}. An $SU(4)$ rotation which rotates the supercharges, $Q' = UQ$, acts on the Clebsch-Gordon matrices as

$$U \lambda^m U^T = R^m{}_n(U) \lambda^m \tag{8.7}$$

where $R^m{}_n$ is an $SO(6)$ rotation matrix. The Clebsch-Gordon matrices λ^m_{ab} are given by the components $(C\Gamma^m)_{ab}$ where Γ^m are the Dirac matrices of

$Spin(5)$ in the Weyl basis satisfying the Clifford algebra $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$, and C is the charge conjugation matrix. The Gamma matrices are given explicitly in terms of Pauli matrices by

$$\Gamma^1 = \sigma_1 \times \sigma_1 \times 1 \quad , \qquad \Gamma^4 = \sigma_2 \times 1 \times \sigma_1 \tag{8.8}$$

$$\Gamma^2 = \sigma_1 \times \sigma_2 \times 1 \quad , \qquad \Gamma^5 = \sigma_2 \times 1 \times \sigma_2 \tag{8.9}$$

$$\Gamma^3 = \sigma_1 \times \sigma_3 \times 1 \quad , \qquad \Gamma^6 = \sigma_2 \times 1 \times \sigma_3, \tag{8.10}$$

where the The charge conjugation matrix is defined by $C\Gamma^m C^{-1} = -\Gamma^{m*}$

$$C = \sigma_1 \times \sigma_2 \times \sigma_2, \quad \Gamma = \sigma_3 \times 1 \times 1, \quad C\Gamma^m = \begin{pmatrix} \lambda^m_{ab} & 0 \\ 0 & \bar{\lambda}^m_{\dot{a}\dot{b}} \end{pmatrix} \tag{8.11}$$

where the un-dotted indices transform in the spinor representation of $Spin(6)$ or the 4 of $SU(4)$ whereas the the dotted indices transform in the conjugate spinor representation of $Spin(6)$ or the $\bar{4}$ of $SU(4)$. The matrices λ^m_{ab} thus defined have the required antisymmetry and transform properties as in (8.7).

8.2 Modular cornucopia

We assemble here together some properties of modular forms, Jacobi forms, and Siegel modular forms.

Modular forms

Let $\mathbb{H}$ be the upper half plane, *i.e.*, the set of complex numbers τ whose imaginary part satisfies $\mathrm{Im}(\tau) > 0$. Let $SL(2,\mathbb{Z})$ be the group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8.12}$$

with integer entries such that $ad - bc = 1$.

A *modular form* $f(\tau)$ of weight k on $SL(2,\mathbb{Z})$ is a holomorphic function on $\mathcal{H}$, that transforms as

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{8.13}$$

for an integer k (necessarily even if $f(0) \neq 0$). It follows from the definition that $f(\tau)$ is periodic under $\tau \to \tau + 1$ and can be written as a Fourier series

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n) q^n\,, \quad q := e^{2\pi i \tau}\,, \tag{8.14}$$

and is bounded as $\mathrm{Im}(\tau) \to \infty$. If $a(0) = 0$, then the modular form vanishes at infinity and is called a *cusp form*. Conversely, one may weaken the growth condition at ∞ to $f(\tau) = \mathcal{O}(q^{-N})$ rather than $\mathcal{O}(1)$ for some $N \geq 0$; then the Fourier coefficients of f have the behavior $a(n) = 0$ for $n < -N$. Such a function is called a *weakly holomorphic modular form*.

The vector space over $\mathbb{C}$ of holomorphic modular forms of weight k is usually denoted by M_k. Similarly, the space of cusp forms of weight k and the space of weakly holomorphic modular forms of weight k are denoted by S_k and $M_k^!$ respectively. We thus have the inclusion

$$S_k \subset M_k \subset M_k^! \,. \tag{8.15}$$

The growth properties of Fourier coefficients of modular forms are known:

1. $f \in M_k^! \Rightarrow a_n = \mathcal{O}(e^{C\sqrt{n}})$ as $n \to \infty$ for some $C > 0$;
2. $f \in M_k \Rightarrow a_n = \mathcal{O}(n^{k-1})$ as $n \to \infty$;
3. $f \in S_k \Rightarrow a_n = \mathcal{O}(n^{k/2})$ as $n \to \infty$.

Some important modular forms on $SL(2,\mathbb{Z})$ are:

1. The *Eisenstein series* $E_k \in M_k$ $(k \geq 4)$. The first two of these are

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240q + \dots\,, \tag{8.16}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504q + \dots\,. \tag{8.17}$$

2. The *discriminant function* Δ. It is given by the product expansion

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 + ... \tag{8.18}$$

   or by the formula $\Delta = (E_4^3 - E_6^2)\,/1728$.

The two forms E_4 and E_6 generate the ring of modular forms, so that any modular form of weight k can be written (uniquely) as a sum of monomials $E_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$. We also have $M_k = \mathbf{C}\cdot E_k \oplus S_k$ and $S_k = \Delta \cdot M_{k-12}$, so that any $f \in M_k$ also has a unique expansion as $\sum_{0\leq n\leq k/12} \alpha_n\, E_{k-12n}\, \Delta^n$ (with $E_0 = 1$). From either representation, we see that a modular form is uniquely determined by its weight and first few Fourier coefficients.

Jacobi forms

Consider a holomorphic function $\varphi(\tau, z)$ from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$ which is "modular in τ and elliptic in z" in the sense that it transforms under the modular group as

(8.19)
$$\varphi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k\, e^{\frac{2\pi i m c z^2}{c\tau+d}}\, \varphi(\tau, z)\,, \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$$

and under the translations of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z}\,, \tag{8.20}$$

where k is an integer and m is a positive integer.

These equations include the periodicities $\varphi(\tau+1, z) = \varphi(\tau, z)$ and $\varphi(\tau, z+1) = \varphi(\tau, z)$, so φ has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n,r)\, q^n\, y^r\ , \qquad\qquad (q := e^{2\pi i\tau},\ y := e^{2\pi i z})\ . \tag{8.21}$$

Equation (8.20) is then equivalent to the periodicity property

(8.22)
$$c(n,r) = C(4nm - r^2;\, r)\,, \qquad \text{where } C(d;r) \text{ depends only on } r \pmod{2m}\,.$$

The function $\varphi(\tau, z)$ is called a *holomorphic Jacobi form* (or simply a *Jacobi form*) of weight k and index m if the coefficients $C(d;r)$ vanish for $d < 0$, *i.e.* if

$$c(n,r) = 0 \qquad \text{unless} \qquad 4mn \geq r^2\ . \tag{8.23}$$

It is called a *Jacobi cusp form* if it satisfies the stronger condition that $C(d;r)$ vanishes unless d is strictly positive, *i.e.*

$$c(n,r) = 0 \qquad \text{unless} \qquad 4mn > r^2\ , \tag{8.24}$$

and conversely, it is called a *weak Jacobi form* if it satisfies the weaker condition

$$c(n,r) = 0 \qquad \text{unless} \qquad n \geq 0 \tag{8.25}$$

rather than (8.23).

Theta functions

In this section, we collect definitions and useful properties of theta function. The Jacobi theta function is defined by

$$\theta[^a_b](v|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(v-b)(n-a)}\,, \tag{8.26}$$

where a, b are real and $q = e^{2\pi i\tau}$. It satisfies the modular properties

$$\theta[^a_b](v|\tau+1) = e^{-i\pi a(a-1)}\theta[^{a}_{a+b-\frac{1}{2}}](v|\tau) \tag{8.27}$$

$$\theta[^a_b]\left(\frac{v}{\tau}|-\frac{1}{\tau}\right) = e^{2i\pi ab+i\pi\frac{v^2}{\tau}}\theta[^a_b](v|\tau) \tag{8.28}$$

The Jacobi-Erderlyi theta functions are the values at half periods,

$$\theta_1(z|\tau) = \theta[^{\frac{1}{2}}_{\frac{1}{2}}](z|\tau), \quad \theta_2(z|\tau) = \theta[^{\frac{1}{2}}_{0}](z|\tau), \quad \theta_3(z|\tau) = \theta[^0_0](z|\tau), \quad \theta_4(z|\tau) = \theta[^{0}_{\frac{1}{2}}](z|\tau) \tag{8.29}$$

In particular,

$$\theta_1(v/\tau, -1/\tau) = i\sqrt{-i\tau}e^{i\pi v^2/\tau}\theta_1(v,\tau) \tag{8.30}$$

The Dedekind η function is defined as

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{8.31}$$

It satisfies the modular property

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \tag{8.32}$$

It is related to the Jacobi-Erderlyi theta functions by the identities

$$\frac{\partial}{\partial v}\theta_1(v)|_{v=0} = 2\pi\ \eta^3(\tau) \tag{8.33}$$

$$\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2\eta^3 \tag{8.34}$$

The partition function of a single left-moving boson is given by

$$Z_{boson}(\tau) := \mathrm{Tr}(q^{L_0}) = \frac{1}{\eta(\tau)}\,. \tag{8.35}$$

Bibliography

[1] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, *Gen.Rel.Grav.* **40** (2008) 2249–2431, [0708.1270].

[2] I. Mandal and A. Sen, *Black Hole Microstate Counting and its Macroscopic Counterpart*, *Nucl.Phys.Proc.Suppl.* **216** (2011) 147–168, [1008.3801].

[3] J. Gomes, *Quantum entropy of supersymmetric black holes*, 1111.2025.

[4] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity*, . San Francisco, USA: Addison-Wesley (2004) 513 p.

[5] R. M. Wald, *General Relativity*, . Chicago, Usa: Univ. Pr. (1984) 491p.

[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, . San Francisco 1973, 1279p.

[7] N. D. Birrell and P. C. W. Davies, *QUANTUM FIELDS IN CURVED SPACE*, . Cambridge, Uk: Univ. Pr. (1982) 340p.

[8] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. vol. 1: Introduction*, . Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[9] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. vol. 2: Loop amplitudes, anomalies and phenomenology*, . Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).

[10] J. Polchinski *String theory. Vol. 1, Cambridge, UK: Univ. Pr.* (1998).

[11] J. Polchinski, *String theory. vol. 2: Superstring theory and beyond*, . Cambridge, UK: Univ. Pr. (1998) 531 p.

[12] A. Einstein *PAW* (1915) 844.

[13] K. Schwarzschild *PAW* (1916) 189.

[14] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D7** (1973) 2333–2346.

[15] S. W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[16] R. M. Wald, *Black hole entropy in the noether charge*, *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038].

[17] V. Iyer and R. M. Wald, *Some properties of noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D50** (1994) 846–864, [gr-qc/9403028].

[18] T. Jacobson, G. Kang, and R. C. Myers, *Black hole entropy in higher curvature gravity*, gr-qc/9502009.

[19] A. Sen, *Entropy Function and AdS(2)/CFT(1) Correspondence*, *JHEP* **11** (2008) 075, [0805.0095].

[20] A. Sen, *Quantum Entropy Function from AdS(2)/CFT(1) Correspondence*, 0809.3304.

[21] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B438** (1995) 109–137, [hep-th/9410167].

[22] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B443** (1995) 85–126, [hep-th/9503124].

[23] A. Sen, *Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and sl(2,z) invariance in string theory*, *Phys. Lett.* **B329** (1994) 217–221, [hep-th/9402032].

[24] A. Sen, *Strong - weak coupling duality in four-dimensional string theory*, *Int. J. Mod. Phys.* **A9** (1994) 3707–3750, [hep-th/9402002].

[25] E. Witten and D. I. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B78** (1978) 97.

[26] A. Dabholkar, D. Gaiotto, and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023, [hep-th/0702150].

[27] S. Banerjee and A. Sen, *Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T**6*, *JHEP* **0803** (2008) 022, [0712.0043].

[28] S. Banerjee and A. Sen, *S-duality Action on Discrete T-duality Invariants*, *JHEP* **0804** (2008) 012, [0801.0149].

[29] S. Banerjee, A. Sen, and Y. K. Srivastava, *Partition Functions of Torsion ¿ 1 Dyons in Heterotic String Theory on T**6*, *JHEP* **0805** (2008) 098, [0802.1556].

[30] A. Dabholkar, J. Gomes, and S. Murthy, *Counting all dyons in N =4 string theory*, 0803.2692.

[31] A. Dabholkar and J. A. Harvey, *Nonrenormalization of the superstring tension*, *Phys. Rev. Lett.* **63** (1989) 478.

[32] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, *Superstrings and solitons*, *Nucl. Phys.* **B340** (1990) 33–55.

[33] R. Dijkgraaf, G. W. Moore, E. P. Verlinde, and H. L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197–209, [hep-th/9608096].

[34] D. Gaiotto, A. Strominger, and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023, [hep-th/0504126].

[35] A. Strominger and C. Vafa, *Microscopic origin of the bekenstein-hawking entropy*, *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029].

[36] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett.* **B391** (1997) 93–98, [hep-th/9602065].

[37] D. Gaiotto, *Re-recounting dyons in N = 4 string theory*, hep-th/0506249.

[38] J. R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072, [hep-th/0605210].

[39] M. C. N. Cheng and E. Verlinde, *Dying dyons don't count*, arXiv:0706.2363 [hep-th].

[40] A. Sen, *Walls of marginal stability and dyon spectrum in N=4 supersymmetric string theories*, *JHEP* **05** (2007) 039, [hep-th/0702141].

[41] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, *Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy*, *JHEP* **12** (2004) 075, [hep-th/0412287].

[42] A. Sen, *Black hole solutions in heterotic string theory on a torus*, *Nucl. Phys.* **B440** (1995) 421–440, [hep-th/9411187].

[43] M. Cvetic and D. Youm, *Dyonic bps saturated black holes of heterotic string on a six torus*, *Phys. Rev.* **D53** (1996) 584–588, [hep-th/9507090].

[44] S. Ferrara, R. Kallosh, and A. Strominger, *N=2 extremal black holes*, *Phys. Rev.* **D52** (1995) 5412–5416, [hep-th/9508072].

[45] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D54** (1996) 1514–1524, [hep-th/9602136].

[46] A. Strominger, *Macroscopic entropy of $n = 2$ extremal black holes*, *Phys. Lett.* **B383** (1996) 39–43, [hep-th/9602111].

[47] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B451** (1999) 309–316, [hep-th/9812082].

[48] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortsch. Phys.* **48** (2000) 49–64, [hep-th/9904005].

[49] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, *Area law corrections from state counting and supergravity*, *Class. Quant. Grav.* **17** (2000) 1007–1015, [hep-th/9910179].

[50] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, *Stationary bps solutions in $n = 2$ supergravity with r**2 interactions*, *JHEP* **12** (2000) 019, [hep-th/0009234].

[51] A. Sen, *Entropy function for heterotic black holes*, hep-th/0508042.

[52] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, *Exact and asymptotic degeneracies of small black holes*, *JHEP* **08** (2005) 021, [hep-th/0502157].

[53] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096, [hep-th/0507014].

[54] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301, [hep-th/0409148].

[55] A. Dabholkar, R. Kallosh, and A. Maloney, *A stringy cloak for a classical singularity*, *JHEP* **12** (2004) 059, [hep-th/0410076].

[56] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A10** (1995) 2081–2094, [hep-th/9504147].

[57] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034, [hep-th/0506176].

[58] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022, [hep-th/0508218].

[59] B. de Wit, S. Katmadas, and M. van Zalk, *New supersymmetric higher-derivative couplings: Full N=2 superspace does not count!*, 1010.2150.

[60] E. Kiritsis, *Introduction to non-perturbative string theory*, hep-th/9708130.

[61] A. Dabholkar and S. Nampuri, *Spectrum of Dyons and Black Holes in CHL orbifolds using Borcherds Lift*, *JHEP* **11** (2007) 077, [hep-th/0603066].

[62] S. Banerjee, A. Sen, and Y. K. Srivastava, *Genus Two Surface and Quarter BPS Dyons: The Contour Prescription*, *JHEP* **03** (2009) 151, [0808.1746].

[63] E. Witten, *Elliptic genera and quantum field theory*, *Commun. Math. Phys.* **109** (1987) 525.

[64] O. Alvarez, T. P. Killingback, M. L. Mangano, and P. Windey, *String theory and loop space index theorems*, *Commun. Math. Phys.* **111** (1987) 1.

[65] S. Ochanine, *Sur les genres multiplicatifs definis par des integrales elliptiques*, *Topology* **26** (1987) 143.

[66] P. H. Ginsparg, *Applied conformal field theory*, hep-th/9108028.