7 (Atmospheric) Planetary Boundary Layer Processes

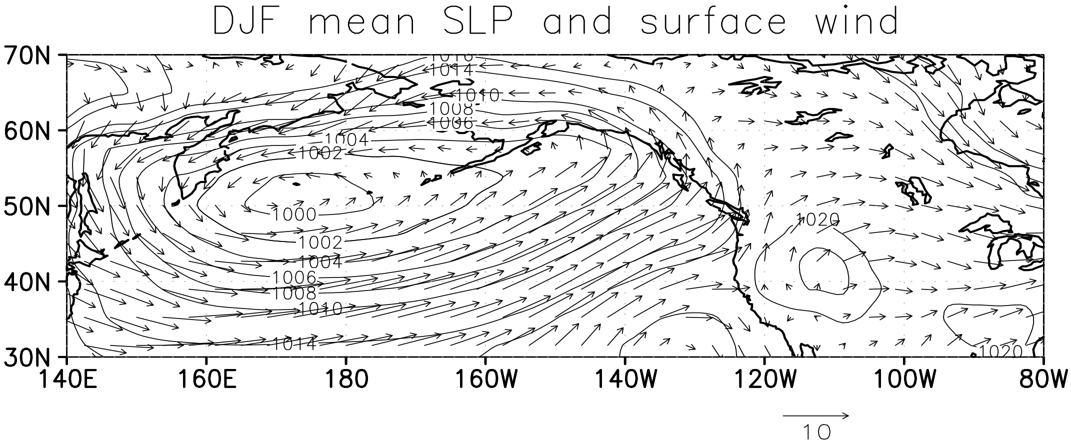

Figure 35: Mean DJF Sea Level Pressure (SLP) and surface winds in the North Pacific from observations (NCEP-NCAR re-analysis). Units are m/s for wind and hpa for SLP.

Figure 35 shows the winter mean (DJF) climatology of Sea Level Pressure (SLP) and surface winds in the North Pacific. As we expect for geostrophy, winds are mainly parallel to the isobars (lines of constant pressure). However, there seems to be a systematic tendency for a component of the winds towards the low pressure. In this section we will try to understand this systematic departure from geostrophy. Do you have a guess why this departure exists?

So far we have ignored the effect of friction on the flows. However, in order to understand climate dynamics it is important to consider the effects of friction, particularly the one provided by the *planetary boundary layer*, which covers roughly the lowest kilometer(s) of the atmosphere. The boundary layer frictional processes are ultimately induced by the molecular viscosity. However, this effect is only relevant in the few millimeters closest to the surface. In the largest part of the planetary boundary layer turbulent eddies take over the role of molecular friction. This is part of the *energy cascade*, meaning that in the large-scale flow ever smaller eddies are embedded that carry energy to ever smaller scales until finally molecular viscosity takes over. As for all sections, a whole lecture course could be devoted to this topic. Therefore we will concentrate on the features that are most relevant and essential to understand climate dynamics.

7.1 The Boussinesq Approximation

The density in the lowest part of the Atmosphere varies little (about 10 % of its mean value). The flow may be considered as essentially incompressible if only the momentum equation is considered. Therefore, for simplicity we assume the density to be constant. Let us consider therefore horizontal momentum equations 132 and 133, in which ρ is considered to be a constant

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \tag{194}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \ . \tag{195}$$

and the continuity equation for incompressible flows 142

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ . \tag{196}$$

This set of equations is equivalent to the barotropic shallow water equations introduced and used several times already. However, the *Boussinesq approximation* goes beyond this, because it also involves an approximation to the vertical momentum equation that is different from the hydrostatic equation and allows for buoyancy effects there (as has been used in the EST course of the last term). Here we do not need to consider this equation.

7.2 Reynolds Averaging

In order to simulate the effect of the smaller scale eddies on the larger scale ("resolved") flow, is it useful to apply an averaging operator to the equations. The idea is that the total flow can be divided into a slow evolving large scale field and into small-scale eddy fluctuations

$$u = \overline{u} + u', \ v = \overline{v} + v' \ . \tag{197}$$

Formally, the operator could be a temporal and/or spatial average. For the total derivative of a quantity A

$$\frac{dA}{dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) A + A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \ , \tag{198}$$

where we have added a zero according to the incompressibility condition 196. Therfeore we may write the total derivative (for incompressible flow!) as

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t} + \frac{\partial Au}{\partial x} + \frac{\partial Av}{\partial y} + \frac{\partial Aw}{\partial z}\right) \ , \tag{199}$$

Application of the averaging operator yields

$$\overline{\frac{dA}{dt}} = \left(\frac{\partial \overline{A}}{\partial t} + \frac{\partial(\overline{A}\overline{u} + \overline{A'u'})}{\partial x} + \frac{\partial(\overline{A}\overline{v} + \overline{A'v'})}{\partial y} + \frac{\partial(\overline{A}\overline{w} + \overline{A'w'})}{\partial z}\right) \ , \tag{200}$$

because

$$\overline{ab} = \overline{(\overline{a} + a')(\overline{b} + b')} = \overline{\overline{a}\overline{b}} + \overline{\overline{a}b'} + \overline{a'\overline{b}} + \overline{a'b'} = \overline{a}\overline{b} + \overline{a'b'} \ ,$$

and $\overline{a'} = \overline{b'} = 0$. Therefore application of the averaging operator to Eqs. 194 and 195 yields

$$\frac{\overline{d}\overline{u}}{dt} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + f\overline{v} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z} \tag{201}$$

$$\frac{\overline{d}\overline{v}}{dt} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} - f\overline{u} - \frac{\partial \overline{v'u'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} - \frac{\partial \overline{v'w'}}{\partial z} \ , \tag{202}$$

where

$$\frac{\overline{d}}{dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z} \tag{203}$$

is the rate of change following the large-scale (or resolved) flow. Applying the zonal average to the continuity equation leads to

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \quad . \tag{204}$$

If we compare Eqs. 201 and 202 with 195 and 195 we see that extra terms emerge if we follow a particle with the average large-scale flow. These can be interpreted as the effects of the small-scale eddies on the large-scale flow, and are called *convergence of eddy momentum fluxes*. Obviousely, in order to solve Eqs. 201 and 202 for the large-scale flow, the additional terms have to be *parameterized* in terms of mean flow properties. This is big topic in fluid dynamics, and is called *closure problem*. Note that a very similar problem occurs when we write down the Navier-Stokes equations for numerical models which intrinsically have a grid (and time) spacing. Turbulence theories give some clues how such parameterizations should look like. One of the simplest one is the *flux-gradient theory*, which states that the effect of small-scale eddies on the large-scale flow is similar to the effect of molecular viscosity on smaller scale flow. The effect of viscosity on the small scale flow is to bring the flow into equilibrium, that is to reduce contrasts or gradient. Also note that the geometry (horizontal surface) means that changes in the vertical direction are much larger than in the horizontal direction (horizontal homogeneity). We have parameterizations of the type:

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z} \tag{205}$$

$$\overline{v'w'} = -K_m \frac{\partial \overline{v}}{\partial z} \ , \tag{206}$$

where K_m can be a function of the vertical coordinate z. All other momentum fluxes can be approximated to be close to zero in Eqs. 201 and 202.

7.3 The Ekman Layer

The Ekman layer is the layer that connects a layer very close to the surface to the free atmosphere where we have near geostrophic equilibrium. Using the geostrophic wind (note that we can consider for the current analysis $f = f_0 = const$)

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}\ , v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}\ ,$$

with this and Eqs. 205-206, the stationary (equilibrium) approximation to Eqs. 201 and 202 are

$$\begin{aligned} K_m\frac{\partial^2 u}{\partial z^2} + f(v - v_g) &= 0 \quad (207)\\ K_m\frac{\partial^2 v}{\partial z^2} - f(u - u_g) &= 0\ , \quad (208) \end{aligned}$$

where we have dropped the overbar for average quantities for convenience (only average quantities appear). Note that also the mean vertical advection term has been dropped because of smallness compared to the other terms. The horizontal advection terms are dropped because of the horizontal homogeneity condition. If we assume $u_g = const.$, and $v_g = const.$ with height, then we can substitute $u_* = u - u_g$ and $v_* = v - v_g$. to get a system of the type

$$K_m\frac{\partial^2}{\partial z^2}\begin{pmatrix} u_* \\ v_* \end{pmatrix} + \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}\begin{pmatrix} u_* \\ v_* \end{pmatrix} = 0 \quad . \quad (209)$$

We assume a solution of the type $u_* = Ae^{imz},\ v_* = Be^{imz}$, then it follows

$$\begin{pmatrix} -K_m m^2 & f \\ -f & -K_m m^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad . \quad (210)$$

Following basic algegra, non-trivial solutions of such a linear are found by setting the determinant of the 2x2 matrix to zero

$$K_m^2 m^4 + f^2 = 0\ . \quad (211)$$

The four solutions are for positive f (northern hemisphere; otherwise we have to use negative f for southern hemisphere)

$$m_1 = \sqrt{i}\sqrt{\frac{f}{k_m}},\ m_2 = -\sqrt{i}\sqrt{\frac{f}{k_m}},\ m_3 = \sqrt{-i}\sqrt{\frac{f}{k_m}},\ m_4 = -\sqrt{-i}\sqrt{\frac{f}{k_m}}\ . \quad (212)$$

With $\sqrt{i} = (1+i)/\sqrt{2}$, we have

$$m_1 = (1+i)\sqrt{\frac{f}{2k_m}},\ m_2 = -(1+i)\sqrt{\frac{f}{2k_m}},\ m_3 = (i-1)\sqrt{\frac{f}{2k_m}},\ m_4 = (1-i)\sqrt{\frac{f}{2k_m}}\ . \quad (213)$$

The boundary conditions are geostrophy ($u = u_g,\ v = v_g$) as z goes to infinity, therefore $u_* = v_* = 0$ and $u = v = 0$ or $v_* = -v_g,\ u_* = -u_g$ at z=0. The boundary condition as z goes to infinity excludes solutions that grow, therefore the solutions with negative i (m_2 and m_4) are excluded. If we insert the solutions m_1 and m_3 back into the original system 210, we can determine the two eigenvectors, which are

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$\mathbf{x_3} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The two eigenvectors are complex conjugate and therefore not independent. Therefore the solution simply given by m_1 ,

$$\begin{aligned} u_* &= ae^{im_1 z} = a[\cos(\gamma z) + i\sin(\gamma z)]e^{-\gamma z} && (214) \\ v_* &= iae^{im_1 z} = ia[\cos(\gamma z) + i\sin(\gamma z)]e^{-\gamma z}\ , && (215) \end{aligned}$$

where we have used $\gamma = \sqrt{f/(2K_m)}$. Let $a = b+ic$, then the real part of the solution is

$$\begin{aligned} u_* &= b\cos(\gamma z)e^{-\gamma z} - c\sin(\gamma z))e^{-\gamma z} && (216) \\ v_* &= -b\sin(\gamma z)e^{-\gamma z} - c\cos(\gamma z)e^{-\gamma z} && (217) \end{aligned}$$

and with $z = 0$: $u_*(z=0) = -u_g = b$, $v_*(z=0) = -v_g = -c$ or

$$\begin{aligned} u &= u_g - [u_g\cos(\gamma z) + v_g\sin(\gamma z)]e^{-\gamma z} && (218) \\ v &= v_g + [u_g\sin(\gamma z) - v_g\cos(\gamma z)]e^{-\gamma z}\ , && (219) \end{aligned}$$

The height of the boundary layer may be defined where the wind is for the first time parallel to the geostrophic wind, which is at $De = \pi/\gamma = \pi\sqrt{2K_m/f}$. We can use this formula to estimate the value of the eddy viscosity K_m. Observations of the mean boundary layer height in mid-latitudes give $De \approx 1$ km, therefore $K_m = 1/2f(De/\pi)^2 \approx 5\ \text{m}^2\ \text{s}^{-1}$. An important application of the Ekman solutions 218 and 219 is that we can calculate the vertical velocity at the top of the Ekman Layer induced by the action of turbulent eddies.

Let us calculate the divergence of the winds in the Ekman Layer

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= \left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)(1 - \cos(\gamma z)e^{-\gamma z}) && (220) \\ &- \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)\sin(\gamma z)e^{-\gamma z} \\ &= -\xi_g\sin(\gamma z)e^{-\gamma z}. \end{aligned}$$

This equation states that the divergence in the Ekman layer is proportional to the negative geostrophic vorticity, a very important effect of the boundary layer. Positive

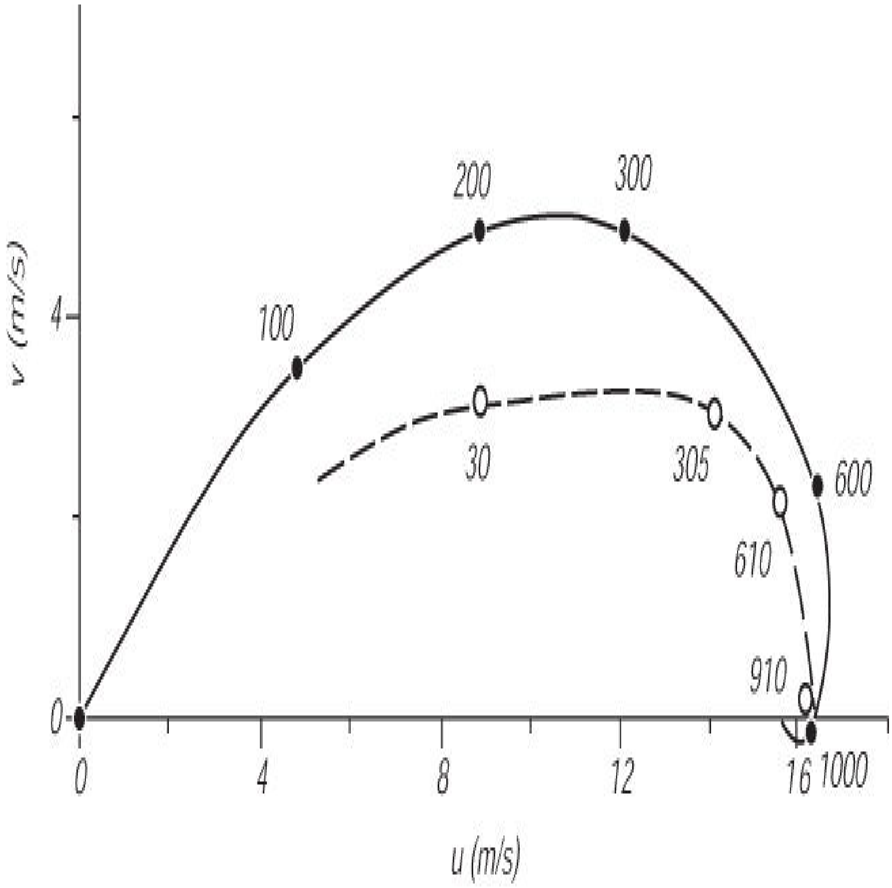

Figure 36: Idealized and observed Ekman Layer velocities Source: http://oceanworld.tamu.edu/resources/ocng_textbook/chapter09/chapter09_02.htm.

(cyclonic vorticity) leads to convergence! Accorgdng to the continuity equation 196, this will lead to vertical motion, which is on top of the Ekman Layer

$$\begin{aligned} w(De) &= -\int_0^{De} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dz \qquad (221) \\ &= \xi_g \, \frac{e^{-\gamma z}}{2\gamma} [\sin(\gamma z) - \cos(\gamma z)] \Big|_0^{De} \\ &= \frac{\xi_g}{2\gamma} \left(1 + e^{-\pi}\right) \\ &\approx \frac{\xi_g}{2\gamma} = \xi_g \sqrt{\frac{K_m}{2f}} \, , \end{aligned}$$

where we have assumed that the geostrophic wind is independent of height within the Ekman Layer. This is again an important result, a positive vortivity leads to upward motion through Ekman effets on top of the boundary layer. This is called *boundary layer pumping* or *Ekman pumping*. It may be used to explain vertical motions and therefore rainfall anomalies induced by the Gill responses in tropical regions as derived in Section 6.3. It states that whenever we calculate a flow response that has (geostrophic) vorticity, this will lead to vertical motion and therefore a rainfall response. Given that geostrophy is valid from approximately 10 degrees away from the equator, this rule can be used for many flow responses. Remember that we have shown in chapter 5 that even the zonal winds in the close equatorial Kelvin waves are in exact geostrophic equilibrium. We can estimate the typical magnitude of the vertical velocity 221 by inserting $\xi_g = 10^{-5}\mathrm{s}^{-1}$, $De = 1km$ or $\gamma = 3 \times 10^{-3}$ m^{-1} to be $w(De) \approx 10^{-5}/(2 \times 3 \times 10^{-3})$ m s^{-1} or 2×10^{-3} m s^{-1}. This a substantial vertical velocity, comparable to the one induced by a heating anomaly of about Q/c_p 1 k/day in the tropical regions, if we use equation 174 and $S_p \approx 5 \times 10^{-4}$ K Pa^{-1} to estimate the vertical velocity:

$$w \approx \frac{Q}{c_p} \frac{1}{S_p \rho g} \quad .$$

Also in the Ocean Ekman Layers exist (have you discussed them?). Clearly at the bottom of the ocean very similar processes take place as discussed here. Even at the top of the oceans we have an Ekman Layer (have you discussed this?). However, the main change is the boundary condition at the surface, which is given by the atmospheric winds that drive the ocean, in the interior the boundary condition can be assumed to be geostrophic again. Otherwise we can use the above derived methodology also to derive the ocean surface Ekman Layer.

Exercises

1. Verify that the Ekman solution 218 and 219 is indeed a solution of the original system of equations 207 and 208.

2. Calculate the scalar product between the pressure gradient and the wind within the Ekman layer given by Eqs. 218 and 219. Is the wind directed into or out of a low pressure system?

3. Write a fortran code that uses the Eqs. 218 and 219 and plot the solution as as phase space diagram (u,v) as in Fig 36. Also, solve the original equations 207 and 208 numerically by keeping the local time derivative in the Ekman equations:

$$\frac{\partial u}{\partial t} = K_m \frac{\partial^2 u}{\partial z^2} + f(v - v_g) \qquad (222)$$

$$\frac{\partial v}{\partial t} = K_m \frac{\partial^2 v}{\partial z^2} - f(u - u_g) \; . \qquad (223)$$

For both analysical and numerical solutions use $K_m = 5$ m^2 s^{-1}, the coriolis parameter at $45°N, u_g = 10$ m s^{-1}, $v_g = 0$. The vertical domain should be [0 m , 3000 m]. Use as initial condition $u = u_g, v = 0$. Compare the numerical stationary with the analytical solution. How long does it take for the solution to become approximately stationary?