4 Baroclinic Instability

4.1 A two-layer Model

Even for a highly idealized mean flow profile the mathematical treatment of baroclinic instability in a continuously stratified atmosphere is rather complicated. Thus, we focus on the simplest model that can incorporate baroclinic processes. The atmosphere is represented by two discrete layers bounded by surfaces numbered 0,2, and 4 (generally taken to be the 0-, 500-, and 1000-hPa surfaces, respectively). The quasi-geostrophic vorticity equation for the midlatitude β plane is applied at levels denoted 1 and 3 and the thermodynamic energy equation is applied at level 2. Before writing the specific equations of the two-layer model, it is convenient to define a *geostrophic streamfunction*, $\psi \equiv \Phi/f_0$ (see definitions leading to Eq. 21). Then the geostrophic wind (Eq. 55) and the geostrophic vorticity (Eq. 66) can be expressed as

$$\mathbf{v} = \mathbf{k} \times \nabla\psi, \quad \xi = \nabla^2\psi \tag{104}$$

The quasi-hydrostatic vorticity equation (68) and the hydrostatic thermodynamic equation (65) can be written with help of (37) in terms of ψ and ω as (assuming no diabatic processes)

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathbf{v}\cdot\nabla(\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = f_0\frac{\partial\omega}{\partial p} \tag{105}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial p}\right) = -\mathbf{v}\cdot\nabla\left(\frac{\partial\psi}{\partial p}\right) - \frac{\sigma}{f_0}\omega \quad . \tag{106}$$

We now apply the vorticity equation (105) at the two levels designated as 1 and 3, which are in the middle of the two layers. To do this we must estimate the divergence term $\partial\omega/\partial p$ at these levels using finite difference approximations to the vertical derivatives

$$\left(\frac{\partial\omega}{\partial p}\right)_1 \approx \frac{\omega_2 - \omega_0}{\delta p}, \quad \left(\frac{\partial\omega}{\partial p}\right)_3 \approx \frac{\omega_4 - \omega_2}{\delta p}, \tag{107}$$

where δp is the pressure interval between levels 0-2 and 2-4 and subscript notation is used to designate the vertical level for each dependent variable. The resulting vorticity equations are

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \mathbf{v}_1\cdot\nabla(\nabla^2\psi_1) + \beta\frac{\partial\psi_1}{\partial x} = f_0\frac{\omega_2}{\delta p} \tag{108}$$

$$\frac{\partial}{\partial t}\nabla^2\psi_3 + \mathbf{v}_3\cdot\nabla(\nabla^2\psi_3) + \beta\frac{\partial\psi_3}{\partial x} = -f_0\frac{\omega_2}{\delta p} \quad , \tag{109}$$

where we have used the fact that $\omega_0 = 0$ and assumed that $\omega_4 = 0$, which is approximately true for a level lower boundary surface. We next write the thermodynamic energy equation (106) at level 2. Here we must evaluate $\partial\psi/\partial p$ using the difference formula

$$(\partial\psi/\partial p) \approx (\psi_3 - \psi_1)/\delta p \quad .$$

The result is

$$\frac{\partial}{\partial t}(\psi_1 - \psi_3) = -\mathbf{v}_2 \cdot \nabla(\psi_1 - \psi_3) + \frac{\sigma \delta p}{f_0}\omega_2 \quad . \tag{110}$$

The first term on the right-hand side in Eq. (110) is the advection of the 250-750 hPa thickness by the wind at 500 hPa. However, ψ_2, the 500 hPa streamfunction, is not a predicted field in this model. Therefore, ψ_2 must be obtained by linearly interpolating between the 250- and 750-hPa levels

$$\psi_2 = (\psi_1 + \psi_3)/2 \quad . \tag{111}$$

If this interpolation formula is used, (108)-(110) become a closed set of prediction equations in the variables ψ_1, ψ_3, and ω_2.

4.2 Linear Perturbation Analysis

To keep the analysis as simple as possible we assume that the streamfunctions ψ_1 and ψ_3 consist of basic state parts that depend linearly on y alone, plus perturbations that depend only on x and t (similar to section 3). Thus, we let

$$\begin{aligned} \psi_1 &= -U_1 y + \psi_1'(x,t) \\ \psi_3 &= -U_3 y + \psi_3'(x,t) \\ \omega_2 &= \omega_2'(x,t) \quad . \end{aligned} \tag{112}$$

The zonal velocities at levels 1 and 3 are then constants with the values U_1 and U_3, respectively. Hence, the perturbation field has meridional and vertical velocity components only. Inserting (112) into (108)-(110) and linearizing yields the perturbation equations (see section 3)

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_1'}{\partial x^2} + \beta\frac{\partial \psi_1'}{\partial x} = f_0\frac{\omega_2'}{\delta p} \tag{113}$$

$$\left(\frac{\partial}{\partial t} + U_3\frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_3'}{\partial x^2} + \beta\frac{\partial \psi_3'}{\partial x} = -f_0\frac{\omega_2'}{\delta p} \tag{114}$$

$$\left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)(\psi_1' - \psi_3') - U_T\frac{\partial}{\partial x}(\psi_1' + \psi_3') = \frac{\sigma \delta p}{f_0}\omega_2' \quad , \tag{115}$$

where we have linearly interpolated to express $\mathbf{v}_2$ in terms of ψ_1 and ψ_3 and have defined

$$U_m \equiv (U_1 + U_3)/2, \quad U_T \equiv (U_1 - U_3)/2 \quad .$$

Thus, U_m and U_T are, respectively, the vertically averaged mean zonal wind and the mean thermal wind for the interval $\delta p/2$. The dynamical properties of this system are more clearly expressed if (113)-(115) are combined to eliminate ω_2'. We first note that (113) and (114) can be rewritten as

$$\left(\frac{\partial}{\partial t} + (U_m + U_T)\frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_1'}{\partial x^2} + \beta\frac{\partial \psi_1'}{\partial x} = f_0\frac{\omega_2'}{\delta p} \tag{116}$$

$$\left(\frac{\partial}{\partial t} + (U_m - U_T)\frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_3'}{\partial x^2} + \beta\frac{\partial \psi_3'}{\partial x} = -f_0\frac{\omega_2'}{\delta p} \quad . \tag{117}$$

We now define the barotropic and baroclinic perturbations as

$$\psi_m \equiv (\psi_1' + \psi_3')/2, \quad \psi_T \equiv (\psi_1' - \psi_3')/2 \tag{118}$$

Adding (116) and (117) and using the definitions in (118) yields

$$\left[\frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x}\right] \frac{\partial^2 \psi_m}{\partial x^2} + \beta \frac{\partial \psi_m}{\partial x} + U_T \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_T}{\partial x^2}\right) = 0 \quad , \tag{119}$$

while subtracting (117) from (116) and combining with (115) to eliminate ω_2' yields

$$\left[\frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x}\right] \left(\frac{\partial^2 \psi_T}{\partial x^2} - 2\lambda^2 \psi_T\right) + \beta \frac{\partial \psi_T}{\partial x} + U_T \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_m}{\partial x^2} + 2\lambda^2 \psi_m\right) = 0 \quad , \tag{120}$$

where $\lambda^2 \equiv f_0^2/[\sigma(\delta p)^2]$. Equations (119) and (120) govern the evolution of the barotropic (vertically averaged) and baroclinic (thermal) perturbation vorticities, respectively. As usual we assume that wavelike solutions exist of the form

$$\psi_m = Ae^{ik(x-ct)}, \quad \psi_T = Be^{ik(x-ct)} \quad . \tag{121}$$

Substituting these assumed solutions into (119) and (120) and dividing through by the common exponential factor, we obtain a pair of simultaneous linear algebraic equations for the coefficients of A, B

$$ik[(c - U_m)k^2 + \beta]A - ik^3 U_T B = 0 \tag{122}$$

$$ik[(c - U_m)(k^2 + 2\lambda^2) + \beta]B - ikU_T(k^2 - 2\lambda^2)A = 0 \quad . \tag{123}$$

From the Mathematical Methods course we know that a homogeneous set of equations has only nontrivial solutions if the determinant of the coefficients for A and B is zero. Thus the phase speed c must satisfy the condition

$$\begin{vmatrix} (c - U_m)k^2 + \beta & -k^2 U_T \\ -U_T(k^2 - 2\lambda^2) & (c - U_m)(k^2 + 2\lambda^2) + \beta \end{vmatrix} = 0 \ , \tag{124}$$

which gives a quadratic dispersion equation in c

$$(c - U_m)^2 k^2 (k^2 + 2\lambda^2) + 2(c - U_m)\beta(k^2 + \lambda^2) + [\beta^2 + U_T^2 k^2 (2\lambda^2 - k^2)] = 0 \quad , \tag{125}$$

The solution for c is

$$c = U_m - \frac{\beta(k^2 + \lambda^2)}{k^2(k^2 + 2\lambda^2)} \pm \delta^{1/2} \quad , \tag{126}$$

where

$$\delta \equiv \frac{\beta^2 \lambda^4}{k^4 (k^2 + 2\lambda^2)^2} - \frac{U_T^2 (2\lambda^2 - k^2)}{(k^2 + 2\lambda^2)} \quad . \tag{127}$$

We have shown that (121) is a solution for the system (119) and (120) only if the phase speed satisfies (126). Although (126) appears to be rather complicated, it is immediately apparent that if $\delta < 0$ the phase speed will have an imaginary part and the perturbations will amplify exponentially. Before discussing the general physical conditions required for exponential growth it is useful to consider two special cases.

As the first special case we let $U_T = 0$ so that the basic state thermal wind vanishes and the mean flow is barotropic. There can be no instability if the thermal wind vanishes (i. e. without horizontal mean-state temperature gradients). The *available potential energy* stored in the mean state temperature gradients is responsible for baroclinic growth! The phase speeds in this case are

$$c_1 = U_m - \beta k^{-2} \tag{128}$$

and

$$c_2 = U_m - \beta(k^2 + 2\lambda^2)^{-1} \tag{129}$$

These are real quantities that correspond to the free (normal mode) oscillations for the two-level model with a barotropic basic state current. The phase speed c_1 is simply the dispersion relationship for a barotropic Rossby wave with no y dependence (see Eq. [75]). Substituting the expression (128) in place of c in (122) and (123) we see that in this case $B = 0$, so that the perturbation is barotropic in structure. The expression (129), on the other hand, may be interpreted as the phase speed of an internal baroclinic Rossby wave. Note that c_2 is a dispersion relationship analogous to the Rossby wave speed for a homogeneous ocean with a free surface, which was given in problem 3 of section 3. But, in the two-level model, the factor $2\lambda^2$ appears in the denominator in place of the f_0/gH for the oceanic case. In each of these cases there is vertical motion associated with the Rossby wave so that static stability modified the phase speed.

Comparing (128) and (129) we see that the phase speed of the baroclinic mode is generally much less than that of the barotropic mode, since for average midlatitude tropospheric conditions $\lambda^2 \approx 2 \times 10^{-12}$ m^{-2}, which is comparable in magnitude to k for zonal wavelength of ~ 4500 km.

Returning to the general case where all terms are retained in (126), the stability criterion is most easily understood by computing the *neutral curve*, which connects all values of U_T and k for which $\delta = 0$ so that the flow is *marginally stable*. From Eq. (126), the condition $\delta = 0$ implies that

$$\frac{\beta^2\lambda^4}{k^4(2\lambda^2 + k^2)} = U_T^2(2\lambda^2 - k^2) \tag{130}$$

or

$$k^4/(4\lambda^4) = 1/2\{1 \pm [1 - \beta^2/(4\lambda^4 U_T^2)]^{1/2}\} \quad . \tag{131}$$

Fig. 4.2 shows nondimensional quantity $k^2/2\lambda^2$, which is a measure of the zonal wavelength, plotted against the nondimensional parameter $2\lambda^2 U_T/\beta$, which is proportional to the thermal wind, according to Eg. (131).

As indicated in the figure, the neutral curve separates the unstable region of the U_T, k plane from the stable region. It is clear that the inclusion of the β effect serves to stabilize the flow, for unstable roots exist only for $|U_T| > \beta/(2\lambda^2)$. In addition to a

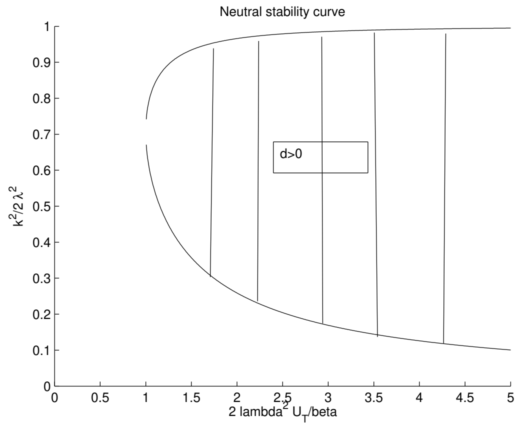

Figure 10: Neutral stability curve for the two-level baroclinic model

minimum value of U_T required for unstable growth depends strongly on k. Thus, the β effect strongly stabilizes the long-wave end of the wave spectrum ($k \to 0$). Again, the flow is always stable for waves shorter than the critical wavelength $L_c = \sqrt{2}\pi/\lambda$ (why?). The long-wave stabilization associated with the β effect is caused by the rapid westward propagation of long waves, which occurs only when the β effect is included in the model.

Differentiating Eq. (130) with respect to k and setting $dU_T/dk = 0$, we find the minimum value of U_T for which unstable waves exist occurs when $k^2 = \sqrt{2}\lambda^2$. This wave number corresponds to the wave of maximum instability. Wave numbers for observed disturbances should be close to the wave number of maximum instability, if U_T were gradually raised from zero the flow would first become unstable for perturbations of wave number $k = 2^{1/4}\lambda$. Those perturbations would then amplify and in the process remove energy from the mean thermal wind, thereby decreasing U_T and stabilizing the flow. Under normal conditions of static stability the wavelength of maximum instability is about 4000 km, which is close to the average wavelength for midlatitude synoptic systems.

Exercises

1. Suppose that a baroclinic fluid is confined between two rigid horizontal lids in a rotating tank in which $\beta = 0$ but friction is presented in the form of linear drag proportional to the velocity (i.e., $\mathbf{Fr} = -\mu\mathbf{v}$). Show that the two-level model perturbation vorticity equations in cartesian coordinates can be written as

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x} + \mu\right)\frac{\partial^2\psi_1'}{\partial x^2} - f_0\frac{\omega_2'}{\delta p} \quad = \quad 0$$

$$\left(\frac{\partial}{\partial t} + U_3\frac{\partial}{\partial x} + \mu\right)\frac{\partial^2\psi'_3}{\partial x^2} + f_0\frac{\omega'_2}{\delta p} \quad = \quad 0 \quad ,$$

where perturbations are assumed in the form given in Eq. (112). The thermodynamic equation remains (115). Assuming solutions of the form (121), show that the phase speed satisfies a relationship similar to (126), with β replaced everywhere by $i\mu k$ and that as a result the condition for baroclinic instability becomes

$$U_T > \mu(2\lambda^2 - k^2)^{-1/2} \quad .$$