3 Rossby Waves

Suggested Literature:

1. Hoskins, B.J. and Karoly D.J., 1981: 'The Steady Linear Response of a Spherical Atmosphere to Thermal and Orographic Forcing', *J. Climate*, **38**, 1179-1196

3.1 Free Barotropic Rossby Waves

The dispersion relation for free barotropic Rossby waves can be derived by linearizing the barotropic vorticity equation in the form (21). This equation states that the absolute (geostrophic) vorticity is conserved following the horizontal (geostrophic) motion. As usual, we assume that the fields can be expressed as small perturbations from a basic state $\psi = \overline{\psi} + \psi'$. We linearize using a basic state that has only flow in zonal direction $\overline{\psi} = -\overline{u}y + const$. This mean state fulfills Eq. (21). With this mean state $\nabla^2\psi = \nabla^2\psi'$. Thus, by linearizing, in the first term the total derivative operator can be replaced by the mean operator and it follows

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \quad . \tag{71}$$

As usual, we seek for solutions of the type

$$\psi' = Ae^{i(kx+ly-\nu t)} \quad . \tag{72}$$

Inserting (72) into (71) yields the dispersion relation

$$(-\nu + k\overline{u})(-k^2 - l^2) + k\beta = 0 \quad , \tag{73}$$

which we can solve immediately for ν

$$\nu = \overline{u}k - \beta k/K^2 \quad , \tag{74}$$

where $K^2 \equiv k^2 + l^2$ is the total horizontal wave number squared. Recalling that $c_x = \nu/k$, we find that the zonal phase speed relative to the mean wind is

$$c_x - \overline{u} = -\beta/K^2 \quad . \tag{75}$$

Thus, the Rossby wave zonal phase propagation is always westward relative to the mean zonal flow. Furthermore, the Rossby wave phase speed depends inversely on the square of the horizontal wave number. Therefore, Rossby waves are dispersive waves whose phases speeds increase rapidly with increasing wavelength. This result is consistent with the discussion in section 2.4, in which we showed that the advection of planetary vorticity, which tends to make the disturbances *retrogress*, increasingly dominates over relative vortivcity advection as the wavelength of a disturbance increases. Equation (75) provides a quantitative measure of this effect in cases where the disturbance is small enough in amplitude.

From Eq. (75) we may calculate the stationary free Rossby wave wavelength

$$K^2 = \beta/\overline{u} \equiv K_s^2 \quad . \tag{76}$$

This means that stationary free Rossby waves only exist if there is a positive mean flow $\overline{u}$. This condition is important for Rossby waves that may be generated by tropical convection.

The group velocity of Rossby waves may be calculated as (exercise!):

$$c_{gx} \equiv \frac{\partial \nu}{\partial k} = \overline{u} + \beta \frac{k^2 - l^2}{K^4} \tag{77}$$

$$c_{gy} \equiv \frac{\partial \nu}{\partial l} = 2\frac{\beta k l}{K^4} \quad . \tag{78}$$

Therefore, the energy propagation of stationary Rossby waves is always eastward (Fig. 8; exercise!).

These waves can also be derived from the original, compressible equations, but the analysis is much more complicated. There are some minor modifications in the phase velocities if the full equations are considered, but the main results remain valid.

3.2 Forced Topographic Rossby waves

Forced stationary Rossby waves are of primary importance for understanding the planetary-scale circulation pattern. Such modes may be forced by longitudinal dependent latent heating, or by flow over topography. Of particular importance for the Northern Hemisphere extratropical circulation are stationary Rossby modes forced by flow over the Rockies and the Himalayas.

As the simplest possible dynamical model of topographic Rossby waves we use the barotropic vorticity equation for a homogeneous fluid of variable depth (e.g. Eqs. 14 or 17). We assume that the upper boundary is at fixed height H and the lower boundary is at the variable height $h_T(x, y)$. We also use the quasi-geostrophic scaling $|\xi| \ll f_0$. Then, from 14 and 17 we have

$$H\frac{d_h(\xi + f)}{dt} = -f_0 \frac{dh_T}{dt} \quad , \tag{79}$$

where is has been also assumed that $h \equiv H - h_t \approx H$ on the left side (i.e. the mountain height is much smaller than the troposphere height). After linearizing (as we did to derive Eq. 71)

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial \psi'}{\partial x} = -\frac{f_0}{H}\bar{u}\frac{\partial h_T}{\partial x} \quad . \tag{80}$$

Lets consider the solutions of Eq. (80) for the special case of a sinusoidal lower boundary. We specify the topography to have the from

$$h_T(x, y) = h_0 \sin(kx + \phi) \cos ly \quad , \tag{81}$$

where ϕ is an arbitrary phase (therefore equivalent to $A\cos kx + B\sin kx$). If we insert the streamfunction perturbation

$$\psi' = \psi_0 \sin(kx + \phi)\cos ly \quad , \tag{82}$$

then Eq. (80) has the steady-state solution (i.e. dropping the partial time derivative) [exercise!]

$$\psi_0 = f_0 h_0 / [H(K^2 - K_s^2)] \quad . \tag{83}$$

The streamfunction is either exactly in phase (ridges over the mountains) or exactly out of phase (troughs over the mountains) with the topography depending on the sign $K^2 - K_s^2$. For long waves, $(K < K_s)$, the topographic vorticity source in Eq. (80) is primarily balanced by meridional advection of planetary vorticity (the β effect). For short waves $(K > K_s)$ the source is balanced primarily by the zonal advection of relative vorticity.

The topographic wave solution (83) has the unrealistic characteristic that when the wave number exactly equals the critical wave number K_s the amplitude goes to infinity. This is the resonant response case when the wave number reaches the stationary wave number of free Rossby waves.

Fig. 8 gives another example for a stationary Rossby wave, caused by ENSO forcing (discuss Eq. 11).

3.3 Turning Latitude

In reality the theory applied here with a constant β and u is a little to over-simplified, and a more correct treatment would make use of the dynamics in spherical coordinates (as in e.g. Hoskins and Karoly, 1981). However, we can derive some properties for an initially north-eastward propagating stationary Rossby wave here knowing that β slowly deceases in the meridional direction. Let us conisder the stationary Rossby wave 85

$$k^2 + l^2 = \beta/\overline{u} \quad . \tag{84}$$

Let's assume a wave generated by ENSO in the tropics moves north-eastward, and that its zonal wave number is a constant. If we further take into account that β decreases to the north with the cosine of the latitude, then the meridional wave number l must decrease until it becomes 0. From this point the wave turns southward again. The latitude in which this occurs is called *turning latitude*, and it is an important property of stationary Rossby waves generated in the tropics. Try to identify the turning latitude in Fig. 8.

We can go a step further, and let also the mean wind u depend on latitude, in which case Eq. 84 has an additional terms:

$$k^2 + l^2 = \frac{\left(\beta - \frac{d^2u}{dy^2}\right)}{\overline{u}} \quad . \tag{85}$$

There are some metric terms missing in this equation, but this expression gives a hint why a strong jet can modify stationary Rossby waves (the full correct expression can be found in Hoskins and Karoly, 1981). Strong jets are therefore also able to modify the turning latitude and other properties of stationary Rossby waves. Fig. 9 shows two examples of stationary wave number distribution in the meridional direction versus the zonal wavenumber for regions with a strong jet (South Asian/Western Pacific region; solid line) and one with a weaker jet (Eastern Pacific region). North of a strong jet the turning latitude is reduced, and we get an effect called *waveguide*, e.g. wave numbers 5 and 6 are essentially trapped in the region between 25° and 35°N. How do you determine the turning latitude in this graph?

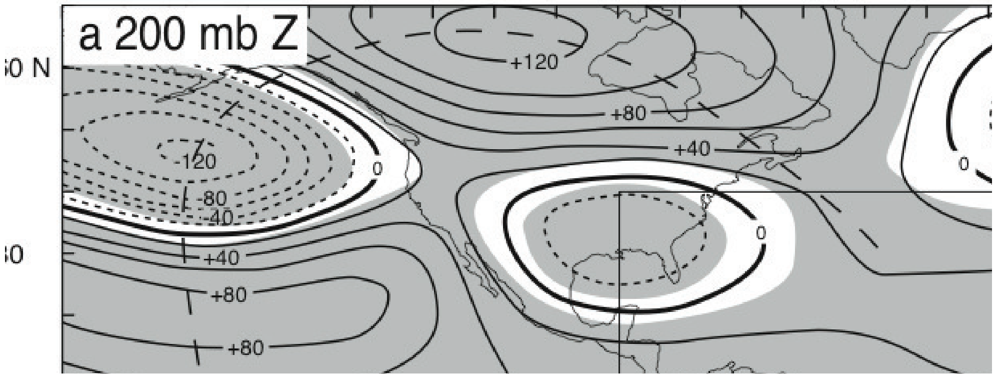

Figure 8: Stationary Rossby wave induced by ENSO.

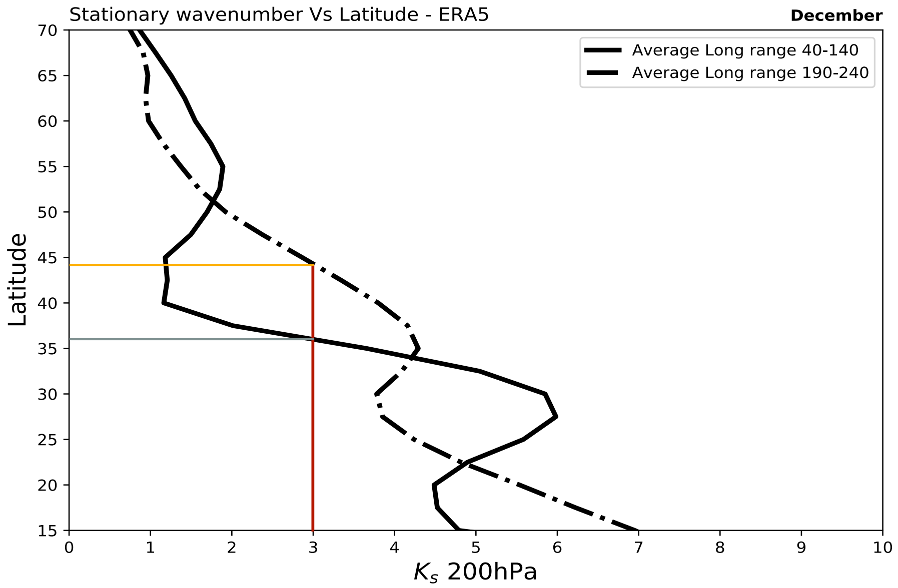

Figure 9: Meridional profile of stationary wave number (K_s). From Master thesis of Alessandro Raganato.

Exercises

1. Derive the group velocities for Rossby waves (77) and (78) and show that for stationary Rossby waves fulfilling Eq. (85), the c_{gx} component is always positive.

2. Show that (83) is the solution of (80) with (81).

3. Using the linearized form of the potential vorticity equation (11) and the β-plane approximation, derive the Rossby wave speed for a homegenous incompressible ocean of depth h. Assume a motionless basic state and small perturbations that depend only on x and t,

$$u = u'(x,t), \quad v = v'(x,t), \quad h = H + h'(x,t) \quad , \tag{86}$$

where H is the mean depth of the ocean. With the aid of the continuity equation for a homogeneous layer

$$\frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} = 0 \tag{87}$$

and the geostrophic wind relationship $v' = gf_0^{-1}\partial h'/\partial x$. Show that the perturbation vorticity equation can be written in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \frac{f_0^2}{gH}\right)h' + \beta\frac{\partial h'}{\partial x} = 0 \tag{88}$$

and that $h' = h_0 e^{ik(x-ct)}$ is a solution provided that

$$c = -\beta(k^2 + f_0^2/gH)^{-1} \quad . \tag{89}$$

If the ocean is 4 km deep, what is the Rossby wave speed at latitude 45° N for a wave of 10000 km zonal wavelength?

4. Rossby-type waves can be generated in a rotating cylindrical vessel if the depth of the fluid is dependent on the radial coordinate. To determine the Rossby wave speed formula for this equivalent β effect, we assume that the flow is confined between rigid lids in an annular region whose distance from the axis of rotation is large enough so that the curvature terms in the equations can be neglected. We then can refer the motion to cartesian coordinates with x directed azimutally and y directed toward the axis of rotation. If the system is rotating at angular velocity Ω and the depth is linearly dependent on y,

$$H(y) = H_0 - \gamma y \quad , \tag{90}$$

show that the perturbation (shallow water) continuity equation ($dH/dt = -H\nabla \cdot \mathbf{v}$) can be written as

$$H_0 \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) - \gamma v' = 0 \tag{91}$$

and that the perturbation quasi-geostrophic vorticity equation is thus

$$\frac{\partial}{\partial t} \nabla^2 \psi' + \beta \frac{\partial \psi'}{\partial x} = 0 \quad , \tag{92}$$

where ψ' is the perturbation geostrophic streamfunction and $\beta = 2\Omega\gamma/H_0$. What is the Rossby wave speed in this situation for waves of wavelength 100 cm in both the x and y directions if $\Omega = 1 s^{-1}$, $H_0 = 20$ cm, and $\gamma = 0.05$? (Hint: Assume that the velocity field is geostrophic except in the divergence term.)

5. Solve the nonlinear potential vorticity conservation equation (19) using (22) and including a Ekman pumping term $r_e \xi, r_e = 1/day$ numerically for a channel with the centre at 45°N of $L_y = 3 \cdot 10^6$ m meridional width and a zonally periodic domains of a length of $L_x = 2 \cdot 10^7$ m using a spatial discretization of $\Delta x = \Delta y = d = 1 \cdot 10^5$ m and a Δt of 1 h. Assume that the top is fixed at a height H = 1.2 $\cdot 10^4$ m, so that the total height of the fluid is given by $h = H - h_t$. Let a sinusoidal mountain be if the shape

$$h_t(x, y) = h_0 \sin(N 2\pi x/L_x) \sin(\pi y/L_y) \quad ,$$

where $h_0 = 1 \cdot 10^3$ m, and let N (the number of the mountain waves in the channel) be a) N=2 and b) N=8. The initial condition zonal flow, which, expressed in streamfunction means

$$\psi(x, y, 0) = -10(y - L_y) \quad .$$

Compare the solutions for a) and b) (after 7 days) and especially compare the position of the eddy streamfunction crests relative to the mountain crests for both cases and interpret the results. Hint: Equation (19), with the condition of a fixed upper height H and given lower topography, including Ekman pumping terms can be written as

$$\frac{\partial}{\partial t}\xi = F(x,y,t) - r_e\xi = -\left[\frac{\partial}{\partial x}(u\xi) + \frac{\partial}{\partial y}(v\xi)\right] - \beta v + \frac{f_0}{h}\left[\frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh)\right] - r_e\xi \quad . \tag{93}$$

and

$$u = -\frac{\partial \psi}{\partial y} \tag{94}$$

$$v = \frac{\partial \psi}{\partial x} \tag{95}$$

$$\xi = \nabla^2 \psi \quad , \tag{96}$$

given that the quasi-geostrophic approximations (15) and (16) are valid. Note that Eq. (93) can be written in its formulation because the geostrophic wind (Eq. 94) is divergence free and the height does not depend explicitly on time. Discretize the terms in Eq. (93) as (using an implicit discretization of the Ekman damping)

$$\xi(t+\Delta t) = \xi(t) + \Delta t F_{i,j}(t) - \Delta t r_e \xi(t+\Delta t) \tag{97}$$

with

$$\begin{aligned} F_{i,j}(t) &= -\frac{1}{2d}\left[(u_{i+1,j}\xi_{i+1,j} - u_{i-1,j}\xi_{i-1,j}) + (v_{i,j+1}\xi_{i,j+1} - v_{i,j-1}\xi_{i,j-1})\right] - \beta v_{i,j} \\ &+ \frac{f_0}{h_{i+1,j}}\frac{1}{2d} \\ &\quad \left[(u_{i+1,j}h_{i+1,j} - u_{i-1,j}h_{i-1,j}) + (v_{i,j+1}h_{i,j+1} - v_{i,j-1}h_{i,j-1})\right] \end{aligned} \tag{98}$$

where the right side is evaluated at the time t, where the fields are already known. This leads to

$$\xi(t+\Delta t) = (\xi(t) + \Delta t F_{i,j})/(1 + \Delta t r_e) \tag{99}$$

Knowing the vorticity, the streamfunction can be determined by Eq. (96), which can be discretized as (see MMG lectures)

$$(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j})/d^2 = \xi_{i,j} \quad . \tag{100}$$

If we can solve this equation for $\psi_{i,j}$, we can derive the velocity fields by using Eqs. (94), (95) in discretized form

$$u_{i,j} = -(\psi_{i,j+1} - \psi_{i,j-1})/(2d) \tag{101}$$

$$v_{i,j} = (\psi_{i+1,j} - \psi_{i-1,j})/(2d) \tag{102}$$

$$\tag{103}$$

A scheme how to solve the initial value problem is:

(a) Given the initial condition $\psi_{i,j}$, (100) can be solved to determine $\xi_{i,j}$, (101) and (102) can be used to determine the velocities

(b) Evaluate $F_{i,j}$ using Eq. (98).

(c) $\xi_{t+\Delta t}$ can be determined by integrating (99).

(d) Knowing $\xi_{t+\Delta t}$, (100) can be inverted to calculate $\psi_{t+\Delta t}$

(e) Go back to (a).

Integrate this scheme for 7 days to reach a steady-state solution and plot the eddy streamfunction and topography.