2 Quasi-geostrophic motion

2.1 The basic equations in isobaric Coordinates

The basic governing equations are (see Eqs. 1 and 2)

The horizontal momentum equations

$$\frac{d\mathbf{v}}{dt} + f\mathbf{k} \times \mathbf{v} = -\frac{1}{\rho}\nabla p \quad , \tag{33}$$

where $\mathbf{v} = \mathbf{i}u + \mathbf{j}v$ and the nabla operator has just the horizontal components.

The vertical equation of motion degenerates for all large-scale motion (e.g. scales more than 100 km) into the hydrostatic equation (discuss how good this approximation is):

$$\frac{\partial p}{\partial z} = -\rho g \quad . \tag{34}$$

Equation (34) states that there is a monotonic relation between pressure p and height z, which leads to the possibility of using p as a vertical coordinate. The basic equation for deriving all transformation from the height to the pressure coordinate system is: $\psi(x, y, p, t) = \psi(x, y, z, t)$, which leads, for example to

$$\frac{\partial \psi}{\partial x} |_p = \frac{\partial \psi}{\partial x} |_z + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} |_p \quad . \tag{35}$$

Inserting $\psi = p$ and applying Eq. (35) also to the derivative in y direction gives the transformation for the horizontal pressure gradient force $\nabla_z p = \rho g \nabla_p z = \rho \nabla_p \Phi$. Thus the horizontal momentum equation reads

$$\frac{d\mathbf{v}}{dt} + f\mathbf{k} \times \mathbf{v} = -\nabla_p \Phi \quad . \tag{36}$$

This looks a little like the horizontal momentum equation of the shallow water model, but it is not! Applying $\psi(x, y, p, t) = \psi(x, y, z, t)$ to a vertical derivative and letting $\psi = p$ gives the hydrostatic equation in pressure coordinates

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\alpha = -\frac{RT}{p} \quad . \tag{37}$$

The total derivative d/dt is invariant and can be expressed as (as follows directly from $\psi = \psi(x, y, p, t)$)

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial p}\frac{dp}{dt} \tag{38}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{39}$$

$$= \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_p + \omega\frac{\partial}{\partial p} \quad . \tag{40}$$

$\omega = dp/dt$ (called the 'omega' vertical velocity) is the pressure change following the motion. Note that when w is positive ω is typically negative.

The Continuity Equation

The easiest way to derive the continuity equation is through the principle of mass conservation. For an infinitesimal mass element we may write:

$$\delta m = \rho \delta V = \rho \delta x \delta y \delta z = -\delta x \delta y \frac{1}{g} \delta p \quad . \tag{41}$$

Note that the first part of equation (41) is just the definition of the density. In the second part the hydrostatic equation (34) has been used to replace the vertical perturbation by a pressure perturbation. Let's calculate the derivative of (41) following the motion (conservation of mass)

$$\frac{1}{\delta m} \frac{d}{dt} \delta m = \frac{g}{\delta x \delta y \delta p} \frac{d}{dt} \frac{\delta x \delta y \delta p}{g} = 0. \tag{42}$$

After appying the product rule of differentiation, and changing the order of differentiation we obtain

$$\frac{1}{\delta x} \delta \frac{d}{dt} x + \frac{1}{\delta y} \delta \frac{d}{dt} y + \frac{1}{\delta p} \delta \frac{d}{dt} p = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0 \ . \tag{43}$$

Letting $\delta x, \delta y, \delta z \rightarrow 0$, it follows the continuity equation in pressure coordianates:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0 \ . \tag{44}$$

In pressure coordinates the full continuity equation takes the form of that of an incompressible fluid, i.e. the time derivative of density does not occur anymore explicitely.

The Thermodynamic Energy Equation

Recall Eqs. 108 or 110 for the Enthalpy, and after multiplying with T from our Earth System Thermodynamics course (for $ds = 0$), which was approximately valid for the atmosphere in which phase transitions from water vapour to liquid water are allowed (do you remember what the symbols L_{lv} and m_v stand for?):

$$c_p \frac{dT}{dt} - \frac{RT}{p} \frac{dp}{dt} = -L_{lv} \frac{dm_v}{dt} \quad . \tag{45}$$

If we further allow diabatic processes to occur (e.g. radiation), then we can simply add Tds/dt on the rhs and abbreviate those terms as Q.

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q \quad . \tag{46}$$

Q is thus the heat added by diabatic processes (i.e. condensation, radiation).

Using the total derivative in pressure coordinates and the definition of ω we have

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)_p - S_p\omega = \frac{Q}{cp} \quad , \tag{47}$$

where the stability factor

$$S_p = \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \tag{48}$$

has been introduced. In Eq. (48) we have used the definition of the potential temperature (Exercise!)

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \quad . \tag{49}$$

p_0 is a constant reference pressure here. Using the dry adiabatic lapse rate $\Gamma_d = g/cp$, we have also (excercise!)

$$S_p = (\Gamma_d - \Gamma)/(\rho g) \quad , \tag{50}$$

where the definition of the lapse rate $-dT/dz = \Gamma$ has been used.

The set of equations (36), (37), (44) and (47) is the basis for our analysis of synoptic-scale motion, but also the basis for many numerical models of the atmospheric circulation.

It is also useful to derive the approximate version of the potential vorticity 31 in pressure coordinates, because it takes a more convenient form. We can write (e.g. using $\psi(x, y, p, t) = \psi(x, y, z, t)$ to evaluate a vertical derivative

$$\frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial p}\frac{\partial p}{\partial z} = -\rho g\frac{\partial \theta}{\partial p} \ . \tag{51}$$

With this 31 becomes

$$q_p = \left[\eta_p \frac{\partial \theta}{\partial p}\right] \quad , \tag{52}$$

where the constant factor $-g$ has been excluded from the definition (this does not matter, why?). Note, that the absolute vorticity $\eta_p = \mathbf{k} \cdot \nabla_p \times \mathbf{v} + f$ is the vertical component of the absolute vorticity and has to be evaluated on pressure levels. The physical interpretation of the approximate potential vorticity in pressure coordinates, q_p, is that a fluid element within an isentropic flow may be considered as confined between two potential temperature values $\Delta\theta$. The thickness of the fluid element, Δp, however, may change. If this occurs, then the absolute vorticity has to adjust in order to maintain potential vorticity conservation.

2.2 Some observed features of the extratropical mean flow

A primary goal of dynamic meteorology is to interpret the observed structure of large-scale atmospheric motions in terms of physical laws governing the motions. These laws, which express the conservation of momentum, mass, and energy completely determine the relationships among the pressure, temperature, and velocity fields. However, the pure laws provide an enormously complicated picture of the motions. For extratropical synoptic-scale motions, however, the horizontal velocities are approximately geostrophic. Such motions, which are often referred to as *quasi-geostrophic*, are simpler to analyze than, for example, tropical disturbances. They are also the major systems of interest in traditional short-range weather forecasting and are thus a reasonable starting point for dynamical analysis. In this section we show that for extratropical synoptic-scale systems the twin requirement of hydrostatic and geostrophic balance constrain the baroclinic motions so that to a good approximation the structure and evolution of the three-dimensional velocity field are determined by the distribution of geopotential height on isobaric surfaces. The equations that express these relationships constitute the quasi-geostrophic system. Before developing this system of equations it is useful to summarize briefly the observed structure of mid-latitude synoptic systems and the mean circulations in which they are embedded.

Zonally averaged cross sections do provide some useful information on the gross structure of the planetary-scale circulation, in which synoptic-scale eddies are embedded. Fig. 4 and 5 show the zonal mean meridional-vertical sections of temperature (left) anomaly from zonal mean and zonal velocity (right) for northern (December-to-February; DJF) and southern winter (June-to-August; JJA), respectively. The vertical direction is measured in pressure (hPa). The average pole-to-equator temperature gradient in the Northern Hemisphere troposphere is much larger in winter than in summer. In the southern hemisphere the difference between summer and winter temperature distributions is smaller, owing mainly to the large thermal inertia of oceans together with the greater fraction of the surface that is covered by oceans in the Southern Hemisphere. The zonal flow and the meridional temperature gradients satisfy to a large degree the thermal wind relation (Exercise!), the largest zonal wind speeds are located in upper levels in regions with the largest meridional temperature gradients

$$\frac{\partial u_g}{\partial p} = \frac{R}{fp}\left(\frac{\partial T}{\partial y}\right)_p \quad . \tag{53}$$

The core of maximum zonal wind speed (called *jet stream* axis) is located just below the *tropopause* (the boundary between troposphere and stratosphere). In both hemispheres the location is about 30°-35° during winter and 40°-45° during summer.

However, there are some important deviations from the zonal mean picture. Fig. 6 shows the zonal wind distribution at the 200 hPa level. As can be seen the largest wind speeds are concentrated just off the coast of Asia and North America, where also the largest meridional temperature gradients occur. Also, whereas the

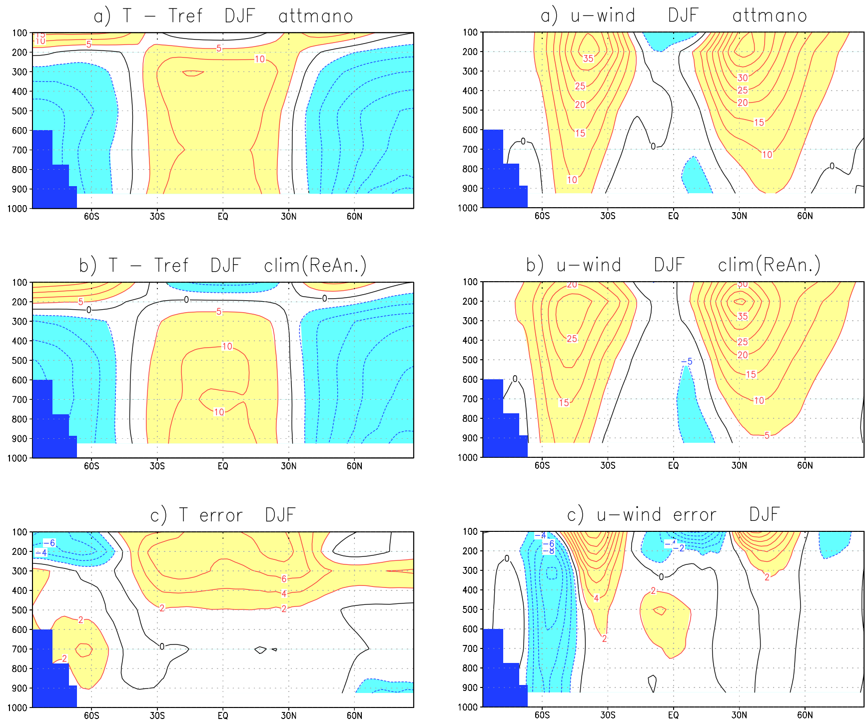

Figure 4: Northern winter meridional-height sections of temperature deviations from zonal mean (left) and zonal wind (right). Units are K for temperature and m/s for wind.

pacific jet is quite zonal, the Atlantic one is clearly tilted from the south-west to the north-east. It is in these regions where most extratropical cyclones and anticyclones develop. I will be shown in section 4 that the mechanisms where these systems draw energy from is the meridional temperature gradient due to an instability called *baroclinic instability*. The systems propagate downstream along the *storm tracks* that approximately follow the jet axis.

The large departure of the northern winter climatological jet stream from zonal symmetry can also be inferred from examination of Fig. 7, which shows the DJF mean 500 hPa geopotential contours (the z from $\Phi = gz$ in Eq. 36). Even after averaging the geopotential height contours for one season, very striking departures from zonal symmetry remain. These are clearly linked to the distribution of continents and Oceans (for example orographic *Rossby waves* due to approximate barotropic potential vorticity conservation [see section 1.3]).

The most prominent asymmetries are the throughs to the east of the American and Asian continents. Referring back to Fig. 6, we see that the intense jet at 35° N and 140° E is a result of the semi-permanent trough in that region (that is the isolines of height show strong gradient in that region).

Thus, it is apparent that the mean flow in which synoptic systems are embedded

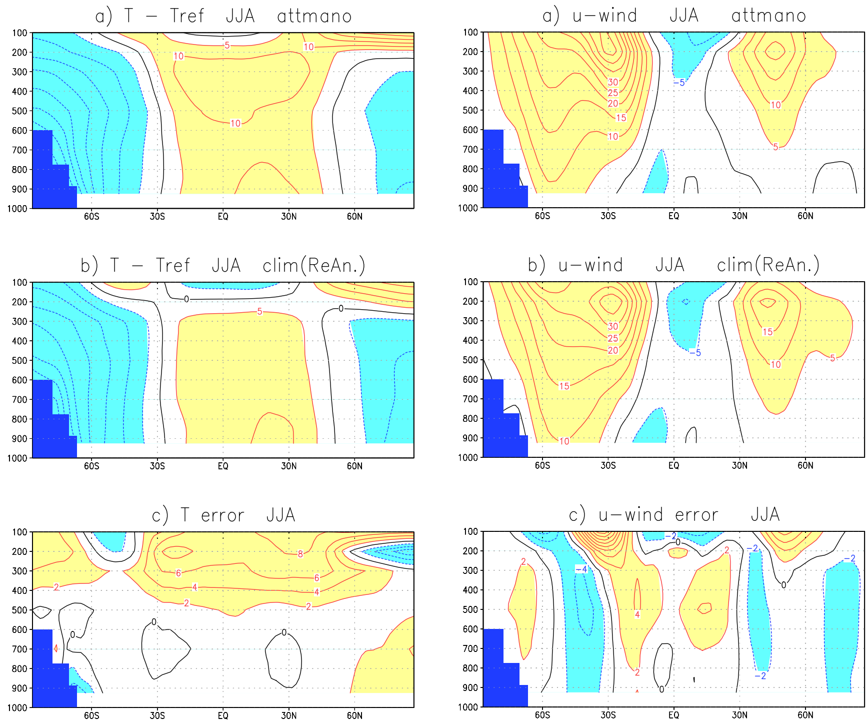

Figure 5: Southern winter meridional-height sections of temperature deviations from zonal mean (left) and zonal wind (right). Units are K for temperature and m/s for wind.

should really be regarded as a longitude-dependent time-averaged flow. In addition to its longitudinal dependence, the planetary-scale flow also varies from day to day owing to its interactions with transient synoptic-scale disturbances.

It is a common observation in fluid dynamics that jets in which strong velocity shears occur may be unstable with respect to small perturbations. By this is meant that any small disturbance introduced into the jets will tend to amplify, drawing energy from the jet as it grows. Most synoptic-scale systems in mid-latitude appear to develop as the result of an instability of the jet-stream flow. This instability, called *baroclinic instability*, depends on the meridional temperature gradient, particularly at the surface. Hence, through the thermal wind relationship, baroclinic instability depends on vertical wind shear.

2.3 The Quasi-geostrophic approximation

The main goal of this chapter is to show how the observed structure of midlatitude systems can be interpreted in terms of the constraints imposed on synoptic-scale motions by the dynamical equations. Specifically we show that for equations that are hydrostatic and nearly geostrophic the three-dimensional flow is determined

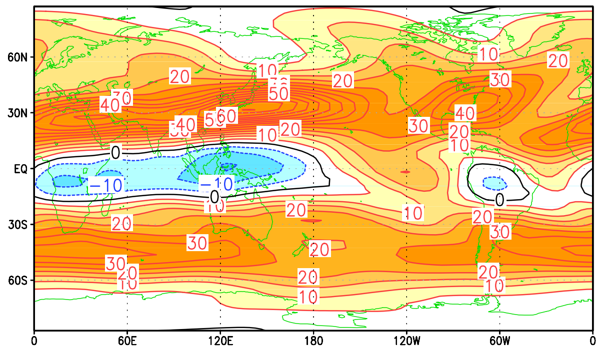

Figure 6: Northern winter (DJF) 200 hPa zonal wind. Units are m/s.

approximately by the isobaric distribution of geopotential height $[\Phi(x, y, p, t)]$. For this analysis, it is convenient to use the isobaric coordinate system both because meteorological measurements are generally referred to constant-pressure surfaces and because the dynamical equations are somewhat simpler in isobaric coordinates than in height coordinates. Thus, use of the isobaric coordinate system simplifies the development of approximate prognostic and diagnostic equations.

2.3.1 Scale Analysis in Isobaric Coordinates

We consider the set of equations (36), (37), (44) and (47). In the following we will drop the notation $()_p$ to indicate derivatives at constant pressure, which is valid in this section for all horizontal and time derivatives. The stability parameter is positive [$S_p \approx 5 \times 10^{-4}$ K Pa^{-1} in the mid-troposphere]. This set of equations, although simplified by use of the hydrostatic approximation and by neglect of some small terms that appear in the complete spherical coordinate form, still contains terms that are of secondary significance for mid-latitude synoptic-scale systems. They can be further simplified by the observation that the horizontal flow is nearly geostrophic and that the ratio of the magnitudes of vertical to horizontal velocities is of the order of 10^{-3}.

We first separate the horizontal velocity into geostrophic and ageostrophic parts by letting

$$\mathbf{v} = \mathbf{v}_g + \mathbf{v}_a \quad , \tag{54}$$

where the geostrophic wind is defined as

$$\mathbf{v}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi \quad , \tag{55}$$

and $\mathbf{v}_a$ is just the difference between the total horizontal wind and the geostrophic wind. Here we have assumed that the meridional scale, L, is small compared to the radius of the earth so that the geostrophic wind may be defined using a constant reference latitude of the Coriolis parameter ($f \approx f_0$ as in equation 15). Note that the definition (55) implies that the geostrophic wind is non-divergent.

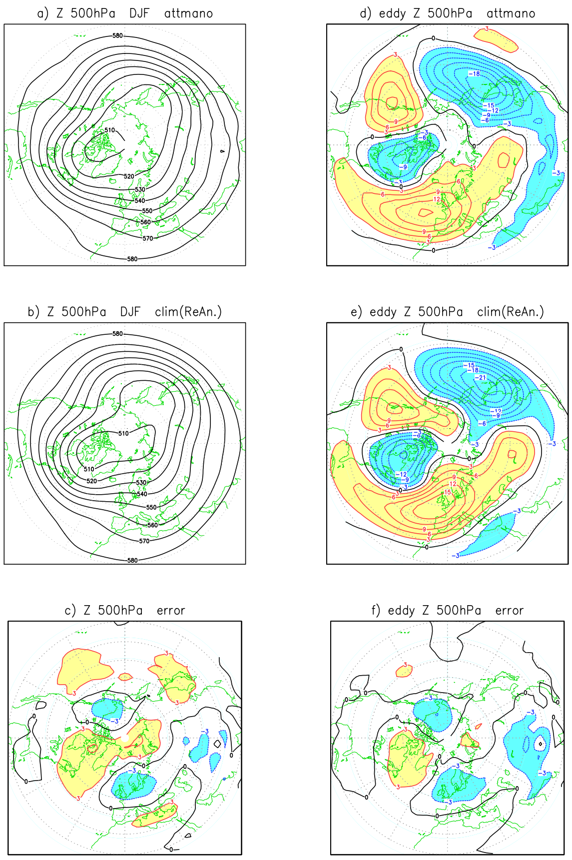

Figure 7: Northern winter (DJF) 500 hPa zonal geopotential height. Units are decametre.

For the systems of interest $|\mathbf{v}_g| \gg |\mathbf{v}_a|$. More precisely,

$$\frac{|\mathbf{v}_a|}{|\mathbf{v}_g|} \sim O(Ro) \approx 10^{-1} \quad . \tag{56}$$

The *Rossby number Ro* has been introduced in Eq. (7).

The momentum can then be approximated to $O(Ro)$ by its geostrophic value, and the rate of change of momentum or temperature following the horizontal motion can be approximated to the same order by the rate of change following the geostrophic wind. Thus, in the total derivative (40), $\mathbf{v}$ can be replaced by $\mathbf{v}_g$ and the vertical advection, which arises from ageostrophic flow, can be neglected. The rate of change of momentum following the total motion is then approximately equal to the rate of change of the geosptrophic momentum following the geostrophic wind:

$$\frac{d\mathbf{v}}{dt} \approx \frac{d_g \mathbf{v_g}}{dt} \quad , \tag{57}$$

where

$$\frac{d_g}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \quad . \tag{58}$$

Note, however, that the vertical advection in the thermodynamic equation, 47, has been combined already with the adiabatic expansion term to provide the stability term $S_p\omega$.

Although a constant f_0 can be used in defining $\mathbf{v}_g$, it is still necessary to retain the dynamical effect of the variation of the Coriolis parameter with latitude in the Coriolis force term in the momentum equation. This variation can be approximated by expanding the latitudinal dependence of f in a Taylor series about a reference latitude ϕ_0 and retaining only the first two terms to yield

$$f = 2|\Omega| \sin\phi \approx f_0 + \beta y \quad , \tag{59}$$

that is the $\sin\phi$-dependence is approximated linearly for a given latitude ϕ_0 by a Taylor series expansion (therefore $\beta = 2|\Omega| cos\phi_0$/a; a being the mean radius of the earth). This approximation is usually referred to as *mid-latitude beta-plane* approximation. For synoptic-scale motions the ratio of the first two terms in the expression of f has the order of magnitude

$$\frac{\beta L}{f_0} \approx \frac{\cos\phi_0}{\sin\phi_0} \frac{L}{a} \sim O(Ro) \ll 1 \quad . \tag{60}$$

This justifies letting the coriolis parameter have a constant value f_0 in the geostrophic approximation and approximating its variation in the coriolis force term by (59).

From Eq. (36) the acceleration following the motion is equal to the difference between the Coriolis force and the pressure gradient force. This difference depends on the departure of the actual wind from the geostrophic wind. We can write, using (54), (59) and (55)

$$\begin{aligned} f\mathbf{k} \times \mathbf{v} + \nabla\Phi &= (f_0 + \beta y)\mathbf{k} \times (\mathbf{v}_g + \mathbf{v}_a) - f_0\mathbf{k} \times \mathbf{v}_g \\ &\approx f_0\mathbf{k} \times \mathbf{v}_a + \beta y \mathbf{k} \times \mathbf{v}_g \quad . \end{aligned} \tag{61}$$

The approximate horizontal momentum equation thus has the form

$$\frac{d_g \mathbf{v_g}}{dt} = -f_0\mathbf{k} \times \mathbf{v}_a - \beta y \mathbf{k} \times \mathbf{v}_g \quad . \tag{62}$$

Since the geostrophic wind (55) is non-divergent, the continuity equation (44) may be written as

$$\nabla \cdot \mathbf{v}_a + \frac{\partial \omega}{\partial p} = 0 \ , \tag{63}$$

which shows that ω is only defined by the ageostrophic part of the wind field (i.e. it is the ageostrophic wind that drives vertical motions that are relevant for energy conversions!!!).

In the thermodynamic energy equation (47) the horizontal advection can be approximated by its geostrophic value. However, as mentioned above, the vertical advection is not neglected, but forms part of the adiabatic heating and cooling term. This term must be retained because the static stability is usually large enough on the synoptic scale that the adiabatic heating or cooling owing to vertical motion is of the same order as the horizontal temperature advection despite the smallness of the vertical velocity. It can be somewhat simplified, though, by dividing the total temperature field T_{tot}, into a basic state (standard atmosphere) portion that depends only on pressure, $T_0(p)$, plus a deviation from the basic state, $T(x, y, p, t)$

$$T_{tot} = T_0(p) + T(x, y, p, t) \quad . \tag{64}$$

Since $|dT_0/dp| \gg |\partial T/\partial p|$ only the basic state portion of the temperature field need to be included in the static stability term and the quasi-geostrophic thermodynamic energy equation may be expressed in the form

$$\frac{\partial T}{\partial t} + \mathbf{v}_g \cdot \nabla T - \left(\frac{\sigma p}{R}\right) \omega = \frac{Q}{cp} \quad , \tag{65}$$

where $\sigma \equiv -RT_0 p^{-1} d \ln \theta_0 / dp$ and θ_0 is the potential temperature corresponding to a basic state temperatue T_0 ($\sigma \approx 2 \times 10^{-6}$ m^2 Pa^{-2} s^{-2} in the midtroposhere).

Equations (62), (55), (37), (63) and (65) constitute the quasi-geostrophic equations. If Q is known these form a complete set in the dependent variables $\Phi, T, \mathbf{v}_g, \mathbf{v}_a$ and ω.

2.4 The Quasi-Geostrophic Vorticity Equation

Just as the horizontal momentum can be approximated to $O(Ro)$ by its geostrophic value, the vertical component of the vortivity can also be approximated geostrophically. Using Eq. (55) the geostrophic vorticity $\xi_g = \mathbf{k} \cdot \nabla \times \mathbf{v}_g$ can be expressed in terms of the Laplacian of the geopotential

$$\xi_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla^2 \Phi \quad . \tag{66}$$

Equation (66) can be used to determine $\xi_g(x, y)$ at any given time from a known field $\Phi(x, y)$. Alternatively, (66) can be solved by inverting the Laplacian operator to determine Φ from a known distribution of ξ provided that suitable conditions on Φ are specified on the boundaries of the region in question. This *invertibility* is one reason why vorticity is such a useful forecast diagnostic; if the evolution of vorticity can be predicted, then inversion of Eq. (66) yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind. Since the Laplacian of a field tends to be a maximum where the function itself is a minimum, positive vorticity implies low values of geopotential and vice versa (see Fig. 7). We will use the invertibility to solve a problem numerically in section 3.

The quasi-geostrophic vorticity equation can be obtained from the x and y components of the quasi-geostrophic momentum equation (62) and yields (exercise!)

$$\frac{d_g \xi_g}{dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g \quad , \tag{67}$$

which should be compared with Eq. (11). Thus the quasi-geostrophic vorticity equation takes the form of the barotropic vorticity equation! Using (63), Equation (67) can be re-written as

$$\frac{\partial \xi_g}{\partial t} = -\mathbf{v}_g \cdot \nabla(\xi_g + f) + f_0 \frac{\partial \omega}{\partial p} \quad , \tag{68}$$

which states that the local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect). The vorticity tendency owing to vorticity advection [the first term on the right in Eq. (68)] may be rewritten as

$$-\mathbf{v}_g \cdot \nabla(\xi_g + f) = -\mathbf{v}_g \cdot \nabla \xi_g - \beta v_g \quad . \tag{69}$$

The two terms on the right represent the geostrophic advections of relative vorticity and the planetary vorticity, respectively. For disturbances in the westerlies, these two effects tend to have opposite signs. In the upstream of a 500 hPa trough, the geostrophic wind is directed from the negative vorticity maximum at the ridge toward the positive vorticity maximum at the trough so that $-\mathbf{v}_g \cdot \nabla \xi_g < 0$. But at the same time, since $v_g < 0$ in that region, the geostrophic wind has its y component directed down the gradient of planetary vorticity so that $-\beta v_g > 0$. Hence, in this region the advection of relative vorticity tends to decrease the local relative vorticity, whereas the advection of planetary vorticity tends to increase the local relative vorticity. Similar arguments (but with reversed signs) apply to a region downstream a trough. Therefore, advection of relative vorticity tends to move the vorticity and trough (and ridge) pattern eastward (downstream). But advection of planetary vorticity tends to move the troughs and ridges westward against the advecting wind field.

The net effect of advection on the evolution of the vorticity pattern depends upon which type of advection dominates. Given a geopotential height wavy field, the vorticity increases with the square of the wave number, so that the first term on the right of Eq. (69) is larger for large wave numbers (i.e. short waves; typically $L_x <$ 3000 km), while for long waves ($L_x >$ 10000 km) the planetary vorticity advection tends to dominate. Therefore, as a general rule, short wavelength synoptic-scale systems should move eastward with the advecting zonal flow while long planetary waves should tend to be stationary of move against the zonal advection. This will be discussed in more details when we derive the dispersion relation for Rossby waves.

Vorticity advection does not alone determine the evolution of meteorological systems. The orographic effects, for example seems to have vanished from Eq. (68). But they are still present, because orography will lead to vertical motions that make the second term on the right important.

Exercises

1. Show that
$$\frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} = (\Gamma_d - \Gamma)/(\rho g)$$
using the definition of potential temperature, and dry adiabatic and actual lapse rates.

2. Show that from (62) follows the quasi-geostrophic vorticity equation (67). [Hint: apply $\partial/\partial x$ to the second component of Eq. (62) and substact $\partial/\partial y$ applied to the first component of Eq. (62)].

3. Derive the thermal wind equation for u-component of the zonal wind (Eq. 53) and also for the v-component in pressure coordinates using the geostrophic relation 55.

4. Suppose that on the 500 hPa surface the relative vorticity at a certain location at 45° N latitude is increasing at a rate of $3 \times 10^{-6}s^{-1}$ per 3 h. The wind is from southwest at 20 m/s. and the relative vorticity decreases toward the northeast at a rate of $4 \times 10^{-6}s^{-1}$ per 100 km. Use the quasi-geostrophic vorticity equation to estimate the horizontal divergence at this location on a β plane.

5. Given the following expression for the geopotential field:
$$\Phi(x, y, p, t) = \Phi_0(p) + f_0[-Uy + k^{-1}V \cos(\pi p/p_0) \sin k(x - ct)] \quad , \qquad (70)$$
where U, V, c, k, p_0 are constants, use the quasi-geostrophic vorticity equation (68) to obtain an estimate for ω. Assume that $df/dy = \beta$ is a constant (not zero) and that ω vanishes for $p = p_0$.