# Carg̀ese Lectures on Black Holes, Dyons, and Modular Forms

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For several $\mathcal{N} = 4$ supersymmetric compactifications, it is now possible to count the dyonic states exactly and compare their degeneracies with the entropy of the corresponding black holes. The dyon partition function is given in terms of a Siegel modular form. For large charges, the logarithm of the degeneracy determined from the partition function is in striking agreement with the Wald entropy of corresponding black holes including subleading corrections. These lectures review this recent progress and discuss a number of puzzles in the physical interpretation of the dyon partition function and their resolution.

## 1. Introduction

One of the important successes of string theory is that one can obtain a statistical understanding of the thermodynamic Bekenstein-Hawking entropy of certain supersymmetric black holes in terms microscopic counting. These black holes satisfy a BPS bound, so their mass $M$ equals the absolute value of a central charge $Z(Q)$ of the supersymmetry algebra. The Bekenstein-Hawking entropy $S_{BH}$ of such a black hole with a charge $Q$ is given by a universal formula [1][2]

$$S_{BH} = \frac{A_H(Q)}{4} \tag{1}$$

where $A_H(Q)$ is the area of the black hole horizon. By Boltzmann's relation this should equal with the statistical entropy $S_{stat} = \log(d(Q))$ where $d(Q)$ is the number of microstates with charge $Q$ that satisfy the BPS bound $M = |Z(Q)|$. In a large number of examples, one finds striking agreement between the two entropies [3]. Such an agreement can be viewed as a nontrivial, nonperturbative consistency check of string theory as a quantum theory of gravity.

In the past few years, the understanding of black hole entropy within string theory has deepened further considerably. For a class of black holes in $\mathcal{N} = 4$ compactifications, it is now possible to compute even the subleading corrections to the Bekenstein-Hawking entropy using Wald's formula. For large charges the subleading corrections are computable and down by powers of $Q$. In these cases, at the next to leading order, there is a full function worth of agreement between $S_{BH}$ and $S_{stat}$ computed using totally different means. These results provide a far more stringent test of string theory as a consistent framework for quantum gravity that goes well beyond the comparison of the leading area law. These lectures describe this recent progress in our understanding of the quantum structure of black holes within string theory.

These notes are organized as follows. CHL compactifications are reviewed in §2 and the dyon partition function in §3. The partition function summarizes the degeneracies of 'quarter-BPS' dyons which preserve one quarter of the supersymmetries. The construction and various representations of the partition functions are summarized in $S4$. The comparison of statistical and thermodynamic entropies for these states in the large charge limit is described in §6. The connection with genus two Riemann surfaces is explained in §7. Various conceptual issues relating to the physical interpretation of the dyon partition function are explained in the subsequent sections. States with negative discriminant are discussed in §8 along with their interpretation as multi-centered configurations in supergravity. The entropy of these multi-centered configuration precisely matches the microscopic predictions. Unlike BPS dyons which preserve half the supersymmetries, quarter-BPS dyons are not stable at all points in the moduli space. After crossing certain lines of marginal stability they can decay into two half-BPS states and disappear from the spectrum. These lines of marginal stability are discussed in §9. The dyon partition function correctly describes the degeneracy of states of given electric and magnetic charges, provided the charges satisfy an irreducibility criterion which is described and derived in §10. A summary of results and conclusions are presented in §11.

## 2. CHL Compactifications

A useful class of string compactifications for our purpose with $\mathcal{N} = 4$ supersymmetry in four dimensions are the CHL orbifolds [4][5]. We recall some basic facts about these compactifications.

To start with, consider heterotic string compactified on $T^6$. This results in a theory with $\mathcal{N} = 4$ supersymmetry in four dimensions. The massless spectrum consists of the supergravity multiplet which contains the metric, a complex scalar field $S = a + ie^{-2\phi}$, and six vector fields– the graviphotons. The axion-dilaton field parametrizes the coset

$$SL(2, \mathbf{Z})\backslash SL(2, \mathbf{R})/SO(2) \tag{2}$$

In addition there are 22 vector multiplets. Each vector multiplet contains a vector field and six real scalars. These parametrize the moduli space

$$SO(22, 6; \mathbf{Z})\backslash SO(22, 6; \mathbf{R})/SO(22) \times SO(6). \tag{3}$$

Altogether there are 28 $U(1)$ gauge fields and hence 28 different electric charges $Q^I_e$ and 28 different magnetic charges $Q^I_m$, $I = 1, \ldots, 28$. The index $I$ transforms in the vector representation of the T-duality group $SO(22, 6; \mathbf{Z})$. We can combine the electric and magnetic charges into a $Q^I_\alpha, \alpha = 1, 2$ with $Q^I_1 = Q^I_e$ and $Q^I_2 = Q^I_m$. The index $\alpha$ transforms as the double of the S-duality group The $SL(2, \mathbf{Z})$,

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \tag{4}$$

with $ad - bc = 1$.

A CHL compactification is obtained by orbifolding the heterotic string compactified on $T^4 \times \tilde{S}^1 \times S^1$ by a $\mathbf{Z}_N$ symmetry generated by $\alpha\beta$, where $\alpha$ is an order N symmetry of the internal CFT of the heterotic string compactified on $T^4$, and $\beta$ is an order $N$ translation along the circle $\tilde{S}^1$. The internal symmetry $\alpha$ has a nontrivial action on the gauge bosons and hence some combinations of bosons are projected out in the orbifolded theory. Because of the order $N$ shift that accompanies $\alpha$, the twisted sectors are massive and no additional gauge bosons arise in the twisted sector. The resulting theory then has gauge group with a rank smaller than 28. Moreover, the symmetry $\alpha$ does not act on the right-moving fermions, and hence fermions are not affected by the orbifold projection. In particular, the orbifold continues to have $\mathcal{N} = 4$ supersymmetry. The CHL orbifolds thus provide us with simple tractable models that have $\mathcal{N} = 4$ supersymmetry and yet a reduced rank $r < 28$. The S-duality group of the model is however smaller and the spectrum of dyons is also different in interesting ways.

The S-duality group of a CHL orbifold is a congruence subgroup $\Gamma_1(N)$ of the $SL(2, \mathbf{Z})$ S-duality symmetry of the original toroidally compactified heterotic string theory. This restriction arises as follows. In the twisted sector, since the orbifolding action includes a $2\pi R/N$ translation along the circle $\tilde{S}$, the twisted states have $1/N$ fractional winding number compared to the winding states in the untwisted sector. We thus have some electric states that are not integrally charged but have $1/N$ integral charge. The states that are S-dual to it are the KK-monopoles associated with the circle $\tilde{S}$ which are integrally charged. This integrality is physically necessitated by the Dirac quantization condition. The S-duality group of the orbifolded theory is thus a subgroup that preserves the Dirac quantization. For example, we have the following transformation.

$$\begin{pmatrix} 1/N \\ n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/N \\ n \end{pmatrix}; \qquad (5)$$

which would preserve Dirac quantization and result in integrally charged KK-monopoles only if $c$ is a multiple of $N$ that is $c = 0\, mod\, N$. This restriction on $SL(2, \mathbf{Z})$ matrices defines the subgroup $\Gamma_0(N)$. A further restriction $a = 1\, mod\, N$ arises from the fact that after S-duality transformation, the resulting state with winding number $1/N$ should be twisted by $\beta$ and not by some other power of $\beta$.

The S-duality group of the orbifolded theory is thus $\Gamma_1(N)$ which consists of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c = 0\, mod\, N, \quad a = 1\, mod\, N \qquad (6)$$

The dyonic degeneracies should be invariant under this S-duality group. This requirement is highly restrictive and results in interesting differences in the dyon spectrum for the various orbifolds.

## 3. The Dyon Partition Function

The partition function of dyons in CHL models is proportional to the inverse of a Siegel modular form of weight $k$ and level $N$ of the group $Sp(2, \mathbf{Z})$ that is naturally associated with a genus two Riemann surface. We now begin by defining some of these concepts below.

Let $\Omega$ be a $(2 \times 2)$ symmetric matrix with complex entries

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \qquad (7)$$

satisfying

$$(\mathrm{Im}\rho) > 0, \ (\mathrm{Im}\sigma) > 0, \ (\mathrm{Im}\rho)(\mathrm{Im}\sigma) > (\mathrm{Im}v)^2 \quad (8)$$

which parametrizes the 'Siegel upper half plane' in the space of $(\rho, v, \sigma)$. It can be thought of as the period matrix of a genus two Riemann surface. For a genus-two Riemann surface, there is a natural symplectic action of $Sp(2, \mathbf{Z})$ on the period matrix. We write an element $g$ of $Sp(2, \mathbf{Z})$ as a $(4 \times 4)$ matrix in the block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (9)$$

where $A, B, C, D$ are all $(2 \times 2)$ matrices with integer entries. They satisfy

$$AB^T = BA^T, CD^T = DC^T, AD^T - BC^T = I \quad (10)$$

so that $g^t J g = J$ where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the symplectic form. The action of $g$ on the period matrix is then given by

$$\Omega \to \Omega' = (A\Omega + B)(C\Omega + D)^{-1}. \qquad (11)$$

The object of our interest is a Siegel modular form $\Phi_k(\Omega)$ It transforms as

$$\Phi_k[\Omega'] = \{\det(C\Omega + D)\}^k \Phi_k(\Omega), \qquad (12)$$

under an appropriate congruence subgroup of level $N$ of $Sp(2, \mathbf{Z})$ [6]. The congruence subgroup $G_0(N)$ is defined in much the same way as the congruence subgroup $\Gamma_0(N)$ of $SL(2, \mathbf{Z})$ as follows. There is a natural embedding of $SL(2, \mathbf{Z}) \sim Sp(2, \mathbf{Z})$ matrices of the for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (13)$$

into $Sp(2, \mathbf{Z})$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix} \in Sp(2, \mathbf{Z}), \quad (14)$$

which along with some other generators generates the full $Sp(2, \mathbf{Z})$. Restricting the entries (13) appropriate for $\Gamma_0(N)$ or $\Gamma_1(N)$, and using the embedding (14) then defines $G_0(N)$ and $G_1(N)$ as a subgroup of $Sp(2, \mathbf{Z})$.

A modular form defined by (12) under $G_0(N)$ transformations we will call a Siegel modular form of 'weight' $k$ and of 'level' $N$. For CHL orbifolds, the partition function corresponds to Siegel modular forms whose weight $k$ is related to the level $N$ by the relation

$$k = \frac{24}{N+1} - 2, \qquad (15)$$

for the cases $N = 1, 2, 3, 5, 7$ [6].

To summarize, for a $\mathbf{Z}_N$ CHL orbifold, the dyonic degeneracies are encapsulated by a Siegel modular form $\Phi_k(\Omega)$ of level $N$ and index $k$ as a function of period matrices $\Omega$ of a genus two Riemann surface. We denote each case by the pair $(N, k)$. For example, the toroidally compactified heterotic string is included in this list as a special case $(1, 10)$.

To see in more detail how the dyon degeneracies are to be extracted from the partition function, let us consider for concreteness the simplest model $(1, 10)$ of toroidally compactified heterotic string as in the original proposal of Dijkgraaf, Verlinde, Verlinde [7]. In this case the relevant modular form is a modular form of weight ten of the full group $Sp(2, \mathbf{Z})$. It is in fact the well-known Igusa cusp form $\Phi_{10}(\Omega)$.

To extract the dyonic degeneracies, note that a dyonic state is specified by the charge vector $Q = (Q_e, Q_m)$ which transforms as a doublet of the S-duality group $SL(2, \mathbf{R})$ and as a vector of the T-duality group $O(22, 6; \mathbf{Z})$. There are three T-duality invariant quadratic combinations $Q_m^2$, $Q_e^2$, and $Q_e \cdot Q_m$ that one can construct from these charges. Given these three combinations, the degeneracy $d(Q)$ of dyonic states of charge $Q$ is then given by

$$d(Q) = g\left(\frac{1}{2}Q_m^2,\ \frac{1}{2}Q_e^2,\ Q_e \cdot Q_m\right), \tag{16}$$

where $g(m, n, l)$ are the Fourier coefficients of $1/\Phi_{10}$,

$$\frac{1}{\Phi_{10}} = \sum_{m \geq -1, n \geq -1, l} e^{2\pi i (m\rho + n\sigma + lv)} g(m, n, l)\,. \tag{17}$$

The parameters $(\rho, \sigma, v)$ can thus be thought of as the chemical potentials conjugate to the integers $\left(\frac{1}{2}Q_m^2,\ \frac{1}{2}Q_e^2,\ Q_e \cdot Q_m\right)$ respectively.

With some modifications, a similar prescription can be given for other more general cases $(k, N)$ [6]. We now turn to the physical properties of the dyon degeneracy defined in this manner.

The degeneracy $d(Q)$ obtained above satisfies a number of physical consistency checks.

- It is integral as expected since it counts the number of states.
- It is S-duality invariant [7][6] with an appropriate prescription [8][9].
- Its logarithm agrees, for large charges, with the Bekenstein-Hawking-Wald entropy of the corresponding black holes to leading and the first subleading order [7][10][6] [11][12].

## 4. Construction and Derivation

The Siegel modular forms are complicated mathematical objects. It turns out that for the modular forms of interest, there exist three different representations which we call Fourier Representation, Product Representation, and Determinant Representation. Each illuminates different aspects of the modular form and is of practical use depending on what physical question one wants to answer. We illustrate these three representations for the case of the Igusa cusp form $\Phi_{10}$.

• *Fourier Representation*

The Siegel modular forms of our interest transform under $Sp(2, \mathbf{Z})$ and is thus naturally associated with a genus two Riemann surface. One way to construct such a genus-two object is by a Maass-Saito-Kurokawa 'lift' of a genus-one object that transforms under $Sp(1, \mathbf{Z}) \sim SL(2, \mathbf{Z}$ [13]. The basic idea is to use the fact that $Sp(2, \mathbf{Z})$ is generated by $Sp(1, \mathbf{Z})$ and some additional generators. In our case for $(k, N) = (1, 10)$, the genus one object has a concrete realization in terms of genus-one theta functions and eta function,

$$\phi_{10}(\rho, v) = \eta^{24}(\rho)\,\frac{\theta_1^2(\rho, v)}{\eta^6(\rho)}. \tag{18}$$

Given the Fourier expansion of this form,

$$\phi_{10}(\rho, v) = \sum_{n \geq 0,\, r \in \mathbf{Z}} a(n, r) q^n y^r, \tag{19}$$

the Maass lift allow one to construct the Fourier representation of the Siegel modular form

$$\Phi_{10}(\rho, v, \sigma) = \sum_{n \geq 0,\, r \in \mathbf{Z}} b(m, n, l) p^m q^n y^r, \tag{20}$$

where the Fourier coefficients $b(m, n, l)$ are determined completely by the Fourier coefficients $a(n, r)$.

This representation is useful for proving the integrality of the dyonic degeneracies $d(Q)$ given in terms of $g(m,n,l)$ in (17) which follows from the integrality of the Fourier coefficients $b(m,n,l)$.

- *Product Representation*

This representation is obtained by "Borcherds' lift" [14] of a genus-on object $\chi(\rho,v)$ given by

$$8[\frac{\theta_2(\rho,v)^2}{\theta_2(\rho)^2}+\frac{\theta_3(\rho,v)^2}{\theta_3(\rho)^2}+\frac{\theta_4(\rho,v)^2}{\theta_4(\rho)^2}]. \tag{21}$$

$$\Phi_{10}(\Omega)=qyp\prod(1-q^ny^lp^m)^{c(4mn-l^2)}, \tag{22}$$

where the coefficients $c(4m-l^2$ are determined by the Fourier coefficients of $\chi(\rho,v)$,

$$\chi(\rho,v)=\sum_{n\geq 0,\, r\in\mathbf{Z}} c(4n-r^2)q^n y^r. \tag{23}$$

The genus-one object $\chi(\rho,v)$ has an interpretation as the elliptic genus of $K3$. As a result, $\Phi_{10}$ is closely related to the elliptic genus symmetric product of $K3$. Note that the symmetric product of $K3$ appears naturally in the context of the $D1D5$ bound states. A D1-brane bound to D5-branes wrapping $K3$ corresponds to an instanton in the worldvolume gauge theory of the D5-branes. The moduli space of instantons for this gauge theory reduces to a symmetric product of $K3$ and the elliptic genus of this space allows one to count the number of bound states of D1D5P [3].

Using this representation, one can then obtain a derivation of the dyon partition function for a class of charges by relating the problem of counting dyons to the problem of counting D1D5P bound states [15][16][17] [18].

- *Determinant Representation*

Finally, the dyon partition function is also closely related to the genus-two chiral partition function of the left-moving heterotic string for the corresponding orbifolds. It is therefore equal to an appropriate product of genus-two determinants of the bosonic fields of the heterotic string. For example, the Igusa cusp form $\Phi_{10}$ is given in terms of the two loop ghost determinants and the determinants of 26 free bosons and is proportional to the product of all ten genus-two theta functions with even spin structure,

$$\Phi_{10}(\Omega)=2^{-12}\prod_{\substack{\alpha,\beta\\ 4\alpha\cdot\beta=even}}\vartheta^2\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\Omega). \tag{24}$$

Note that the here $\alpha$ and $\beta$ are 2-dimensional row vectors with entries either 0 or $1/2$ corresponding to the various choices of spin structures.

The connection to genus-two Riemann surfaces follows by relating the dyons to various configurations of the genus two worldvolume of $K3$-wrapped M5-brane by using string webs and M-theory lift. Since the $K3$-wrapped M5-brane is the heterotic string, the problem of counting these dyonic configurations then reduces to a worldsheet computation in perturbative string theory of the genus-two partition function [19][20]. This representation makes the modular properties manifest since the $Sp(2,\mathbf{Z})$ is naturally associated with the genus-two worldsheet. The appearance of the genus-two Riemann surface will be described in the §7.

## 5. S-Duality Invariance

The prescription for extracting $d(Q)$ from (17) and (16) is not quite complete. The first physical requirement on the degeneracy $d(Q)$ given by (16) is that it should be invariant under the S-duality group of the theory. A naive application of (17) and (16) reveals that the resulting degeneracy is not S-duality invariant. Hence we need to refine the prescription to correctly take into account this physical requirement.

To illustrate the point by explicit Fourier expansion, let us look at states with

$$\frac{1}{2}Q_m^2=-1,\quad \frac{1}{2}Q_e^2=-1,\quad Q_e\cdot Q_m=N. \tag{25}$$

Then according to (16), the degeneracy of such states can be read off from the coefficient of $y^N/qp$ in the Fourier expansion (17). From the product representation of $\Phi_{10}$ given for example in [7], we see that we need to pick the term that goes as $p^{-1}q^{-1}y^N$ in the expansion of

$$\frac{1}{qp(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2}=\sum_{N=1}^{\infty}Nq^{-1}p^{-1}y^N \tag{26}$$

which implies that

$$d(-1,-1,N) = N. \tag{27}$$

Let us now look at what is required for invariance under $SL(2,\mathbf{Z})$ transformations. Consider, for example, the element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{28}$$

of the S-duality group which takes $(Q_e, Q_m)$ to $(Q_m, -Q_e)$. Hence $(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e \cdot Q_m)$ goes to $(\frac{1}{2}Q_e^2, \frac{1}{2}Q_m^2, -Q_e \cdot Q_m)$. Invariance of the spectrum under this element of the S-duality group would then predict $d(-1,-1,-N) = d(-1,-1,N) = N$. However, from the expansion (26) we see that there are no terms in the Laurent expansion that go as $y^{-N}$ and hence an application of the formulae (16) and (17) would give $d(-1,-1,-N) = 0$ in contradiction with the prediction from S-duality.

This apparent lack of S-duality invariance is easy to fix with a more precise prescription. Note that the function $(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{-2}$ has a $\mathbf{Z}_2$ symmetry generated by the element $S$ of the S-duality group that takes $y$ to $y^{-1}$. Under this transformation the contour $|y| < 1$ is not left invariant but instead gets mapped to the contour $|y| > 1$. The new contour cannot be deformed to the original one without crossing the pole at $y = 1$ so if we are closing the contour around $y = 0$ then we need to take into account the contribution from this pole at $y = 1$. Alternatively, it is convenient to close the contour at $y^{-1} = 0$ instead of $y = 0$. Then we do not encounter any other pole and because of the symmetry of the function $(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{-2}$ under $y$ going to $y^{-1}$, the Laurent expansion around $y$ has the same coefficients as the Laurent expansion around $y^{-1}$. We then get,

$$\frac{1}{pq(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2} = \sum_{N=1}^{\infty} N p^{-1} q^{-1} y^{-N}. \tag{29}$$

If we now define $d(-1,-1,-N)$ as the coefficient of $qpy^{-N}$ in the expansion (29) instead of in the expansion (26) then $d(-1,-1,-N) = N = d(-1,-1,N)$ consistent with S-duality.

States with negative $N$ must exist if states with positive $N$ exist, not only to satisfy S-duality invariance but also to satisfy parity invariance. The $\mathcal{N} = 4$ super Yang-Mills theory is parity invariant. Under parity, our state with positive $N$ goes to a state with negative $N$ and the asymptotic values $\chi$ of the axion also flips sign at the same time. Hence if a state with $N$ positive exists at $\chi = \chi_0$ then a state with $N$ negative must exist at $\chi = -\chi_0$. Thus, the naive expansion (26) would give an answer inconsistent with parity invariance and one must use the prescription we have proposed, to satisfy parity invariance. Note that even though S-duality and parity both take the states $(-1,-1,N)$ to $(-1,-1,-N)$ they act differently on the moduli fields.

The main lesson of the above example is that we need to use different points of expansion for different charges. Equivalently, we need to use different contours for the inverse Fourier transform for different charges to obtain $d(Q)$ from the partition function. The function $1/\Phi_{10}$ has many more poles in the $(\rho, \sigma, v)$ space at various divisors that are the $Sp(2,\mathbf{Z})$ images of the pole at $y = 1$ and in going from one contour to the other these poles will contribute. A practical way to state the prescription is to define the degeneracies $d(Q)$ by formulae (16) and (17) first for charges that belong to the 'fundamental cell' in the charge lattice satisfying the condition $\frac{1}{2}Q_m^2 \geq -1$, $\frac{1}{2}Q_e^2 \geq -1$, and $Q_e \cdot Q_m \geq 0$. For these charges $d(Q)$ can be represented as a contour integral for a contour of integration around $p = q = y = 0$ that avoids all poles arising as images of $y = 1$. This can be achieved by allowing $(\rho, v, \sigma)$ to all have a large positive imaginary part as noted also in [18]. For other charges, the degeneracy is defined by requiring invariance under $SL(2,\mathbf{Z})$. The degeneracies so defined are manifestly S-duality invariant. This statement of S-duality invariance might appear tautologous, but its consistency depends on the highly nontrivial fact that an analytic function defined by $\Phi_{10}(\rho, \sigma, v)$ exists that is $SL(2,\mathbf{Z})$ invariant. Its pole structure guarantees that one gets the same answer independent of which way the contour is closed.

Let us now see this in more detail for general charges. Inverting the relation (17) we can write

$$d(Q) = \int_{\mathcal{C}} d^3\Omega \, e^{-i\pi Q'^t \cdot \Omega \cdot Q} \, \frac{1}{\Phi_{10}(\Omega)} \tag{30}$$

where the integral is over the contours

$$0 < \mathrm{Re}(\rho) \le 1,\ 0 < \mathrm{Re}(\sigma) \le 1,\ 0 < \mathrm{Re}(v) \le 1 \quad (31)$$

for the real parts of the three coordinates $(\rho, \sigma, v)$. All coordinates also have fixed and large imaginary part to avoid poles such as the pole at $y = 1$. This defines the integration curve $\mathcal{C}$ as a 3-torus in the Siegel upper half plane.

We would like to now give a prescription for the degeneracy (16) that is invariant under an S-duality transformation

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \to \begin{pmatrix} Q_e \\ Q_m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_e \\ Q_m \end{pmatrix}. \quad (32)$$

The degeneracy of charges $Q'$ related to $Q$ by an S-duality transformation are defined by

$$d(Q') = \int_{C'} d^3\Omega\, e^{-i\pi Q'^t \cdot \Omega \cdot Q'} \frac{1}{\Phi_{10}(\Omega)} \quad (33)$$

where $C'$ is chosen to be related to $C$ by a duality transformation as follows. We define

$$\Omega' \equiv \begin{pmatrix} \rho' & v' \\ v' & \sigma' \end{pmatrix} = (A\Omega + B)(C\Omega + D)^{-1}, \quad (34)$$

We can change the integration variable from $\Omega$ to $\Omega'$. Using these transformation properties and the modular properties of $\Phi_{10}$ we see that

$$\begin{aligned} d^3\Omega' &= d^3\Omega\,, && (35) \\ \Phi_{10}(\Omega') &= \Phi_{10}(\Omega)\,, && (36) \\ Q'^t \cdot \Omega' \cdot Q' &= Q^t \cdot \Omega \cdot Q && (37) \end{aligned}$$

If we choose the integration contour $\mathcal{C}'$ as the one that maps to $\mathcal{C}$ under the duality transformation on the integration variables (34) then that ensures duality invariance of the degeneracies. We therefore conclude

$$d(Q') = \int_{C'} d^3\Omega'\, e^{-i\pi Q'^T \cdot \Omega' Q'} \frac{1}{\Phi_{10}(\Omega')} = d(Q)\,. \quad (38)$$

## 6. Comparison with Wald Entropy

To compute the microscopic, statistical entropy $S_{stat} = \log(d(Q)$ for large charges, one needs to know the asymptotic behavior of $d(Q)$. The degeneracies $d(Q)$ defined by (16) can be obtained by inverse Fourier transform of partition function (17). From this integral representation, the asymptotics in the limit $Q_e^2$ and $Q_m^2$ are both large and of the same order can be worked out by saddle point evaluation.

On the other hand, to compute the macroscopic, thermodynamic entropy $S_{BH}$ of the black hole, one needs to know the low energy effective action. The leading Bekenstein-Hawking entropy is obtained from the leading two-derivative supergravity action that contains the Einstein-Hilbert term. The subleading corrections to the entropy that we discuss below can be computed by including the four derivative terms of the low-energy string effective action in the analysis and then using Wald's generalization of the entropy formula.

For the details of the computation we refer the reader to [10][6][18], and summarize the results below.

One finds that both microscopic and macroscopic entropy computed from the two very different sources mentioned above are given by the extremum value of the *same* 'entropy function' $F(a, S)$. This function depends on two real variables $a$ and $S$, which are the asymptotic values of the axion $a$ and the dilaton $S = e^{-2\phi}$ and is given by

$$\begin{aligned} F(a,S) \;=\; & \frac{\pi}{2}\Big[\frac{a^2+S^2}{S}Q_m^2 + \frac{1}{S}Q_e^2 \\ & -2\,\frac{a}{S}\,Q_e \cdot Q_m + 128\,\pi\,\phi(a,S)\Big] \\ & + \text{constant} + \mathrm{O}(\mathrm{Q}^{-2})\,, \qquad (39) \end{aligned}$$

where

$$\phi(a,S) \equiv -\frac{1}{64\pi^2}\Big\{(k+2)\ln S + \ln f^{(k)}(a+iS) + \ln f^{(k)}(a+iS)^*\Big\} \quad (40)$$

with

$$f^{(k)}(\tau) \equiv \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}\,. \quad (41)$$

It is remarkable how and to what level of detail the two entropies agree with each other. It is essential that all factors in the four derivative action and various normalizations work out just right for this to work. This agreement attests to the beautiful and highly nontrivial internal consistency between the microscopic and macroscopic structure of string theory. The two entropies are computed by totally different means and are *a priori* independent of each other. It is only within the context of string theory, that they can be viewed as two different descriptions of the same object namely the dyonic black hole and hence are expected to agree.

## 7. Genus-Two Riemann Surfaces

One puzzling feature of the proposed dyon partition function is the appearance of the discrete group $Sp(2, \mathbf{Z})$. This group has no direct physical significance because it cannot be viewed as a subgroup of the U-duality group. Moreover, both $Sp(2, \mathbf{Z})$ and the period matrix $\Omega$ are objects that are naturally related to a genus-two Riemann surface. It is equally puzzling why a genus-two surface should play a role in the counting of dyons. An explanation for these puzzles can be provided following the reasoning in [19] which we review below.

Let us first consider the toroidally compactified theory. Heterotic on $T^4 \times T^2$ is dual to Type-IIB on $K3 \times T^2$. The $SL(2, \mathbf{Z})$ symmetry which is the electric-magnetic S-duality group in the heterotic description maps to a geometric T-duality group in the IIB description that acts as the mapping class group of the $T^2$ factor. As a result, the $A$-cycle of the $T^2$ corresponds to electric states and the $B$-cycle to the magnetic states. Now, Type-IIB compactified on a small $K3$ has an effective 1-brane which is a bound state of D5, NS5 branes wrapped on K3, D3 branes wrapped on the 22 2-cycles of the $K3$ and D1,F1 strings. A half-BPS state that is purely electric in the heterotic description then corresponds to the 1-brane wrapping on the $A$-cycle of the torus in the IIB description. The magnetic dual of this state corresponds to the same 1-brane wrapping the $B$-cycle. The 1-brane can in general carry left-moving oscillations.

A quarter-BPS dyon is a bound state of electric and magnetic charges associated with different $U(1)$ gauge fields. In the Type-IIB theory, this bound state is described as a string web wrapping the torus [21][22]. The web has two vertices and at each vertex there is three-string junction as shown in Fig. 1. At the three string junction, a 1-brane carrying charge $Q_e$ (shown in red) combines with the other carrying charge $Q_m$ (shown in blue) to form a 1-brane with charge $Q_e + Q_m$ (shown in green). The simplest such example to keep in mind is to take in the IIB picture $Q_e$ to be a $K3$-wrapped D5-brane, $Q_m$ to be the $K3$-wrapped NS5-brane so that $Q_e + Q_m$ is a $K3$-wrapped $(1, 1)$ 5-brane.

To compute the partition function, we compactify time on a Euclidean circle with supersymmetric boundary conditions. Since Type-IIB string on a circle is dual to M-theory on a $T^2$, we have a compactification of Euclidean M-theory on $K3 \times T^2 \times T^2$. Under this duality, NS5 and D5 branes of IIB map to the M5-brane and circle-wrapped D3-branes to M2-branes. In M-theory all 5-branes are represented by a single M5-brane wrapping $K3$. The topology of the quarter-BPS web of the effective 1-brane wrapping a 2-torus with two vertices is that of a genus two Riemann surface $\Sigma_2$ after adding the Euclidean time circle. Thus the low-energy description of a quarter-BPS brane is a Euclidean M5-brane wrapping $K3 \times \Sigma_2$ embedded in $K3 \times T^4$. This embedding is achieved by the natural holomorphic embedding of a genus two Riemann surface into a $T^4$ given by the Abel map which maps a complex curve into its Jacobian. The dyonic partition function is then given by the sum over all 'left-moving' fluctuations of this worldvolume. Since K3-wrapped M5-brane is the heterotic string we are thus led to computing the genus-two partition function of the left-moving fluctuations of the heterotic string.

Consistent with this picture it is known that the Igusa cusp form $\Phi_{10}$ that appears in the dyon partition function in this case with $N = 1$ is precisely the left-moving genus-two partition function of the toroidally compactified heterotic string. Note that at genus two, the ghost determinants and the light-cone directions do not quite cancel out. Thus, the fact that one obtains a Siegel form of weight 10 depends nontrivially on the ghosts and the light-cone directions and does not follow merely from the 24 transverse directions.

So far we have been discussing the toroidally compactified case with $N = 1$. For CHL orbifolds

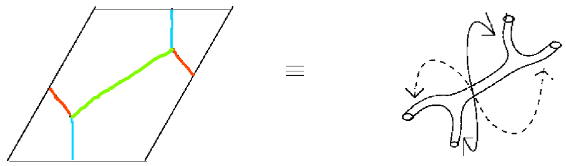

Figure 1. A dyon can be represented as a string web on a torus which in M-theory looks like a genus two Riemann surface.

for other values of $N$, the genus-two worldsheet would have a branch cut of order $N$ along one of the cycles. This can be seen most easily in the string web picture. The CHL orbifold action combines an order $N$ shift $\beta$ along one of the circles with an order $N$ left-moving twist $\alpha$ of the internal CFT. This implies that to construct a state wrapping a compact $T^2$ from the string web, two ends of the web in the fundamental cell of the $T^2$ along one of the cycles (shown in blue for example in Fig. 1) are joined after an order $N$ twist $\alpha$. Using M-theory lift and heterotic dual as above, the resulting genus-two worldsheet then has a branch cut across which some of the left-moving fields undergo a $\mathbf{Z}_N$ twist.

To compute the genus-two partition function in the orbifolded theory, it is necessary to compute twisted determinants for left-moving bosons of the heterotic string. These are in general hard computations but in the case of $\mathbf{Z_2}$ orbifolds, using various CFT techniques one can obtain explicit expression for the genus-two partition function [20]. It precisely equals the correct Siegel modular form $\Phi_6(\Omega)$ as expected, confirming the consistency of this picture.

## 8. States with Negative Discriminant

The comparison above presupposes that for a given a BPS dyonic state with electric and magnetic charges $(Q_e, Q_m)$ it is possible to find a supersymmetric black hole solution of the effective action with the same charges and mass and then compute its entropy. This is not always possible. To see this, it is useful to define the 'discriminant' $\Delta$ by

$$\Delta(Q) = Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2. \tag{42}$$

which is the unique quartic invariant of the full U-duality group $SO(22, 6; \mathbf{Z}) \times SL(2, \mathbf{Z})$. For a black hole with charges $(Q_e, Q_m)$, the attractor value of the horizon area is proportional to the square root of the discriminant and the entropy is given by

$$S(Q) = \pi \sqrt{\Delta(Q)}\ . \tag{43}$$

On the microscopic side also, the discriminant is a natural quantity. It is useful to think of $SL(2, \mathbf{Z})$ as $SO(1, 2; \mathbf{Z})$ which has a natural embedding into $Sp(2, \mathbf{Z}) \sim SO(2, 3; \mathbf{Z})$. The dyon degeneracy formula depends on the T-duality invariant vector of $SO(1, 2; \mathbf{Z})$

$$\begin{pmatrix} Q_m^2/2 \\ Q_e^2/2 \\ Q_e \cdot Q_m \end{pmatrix} \tag{44}$$

The discriminant is the norm of this vector with the Lorentzian metric

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{45}$$

With this norm, for a given state $(Q_m^2/2,\ Q_e^2/2,\ Q_e \cdot Q_m)$,the vector (44) is spacelike, lightlike, or timelike depending on whether $\Delta$ is positive, zero, or negative. We can accordingly refer to the state as spacelike, lightlike, or timelike.

Clearly, to obtain a physically sensible, nonsingular, supersymmetric, dyonic black hole solution in supergravity, it is necessary that the discriminant defined in (42) is positive and large so that the entropy defined in (43) is real. The vector in (44) in this case is spacelike. This fact seems to lead to the following puzzle regarding the dyon degeneracy formula. The formula is valid for a large number of states that can have vanishing or negative discriminant and would in fact predict nonvanishing degeneracy for such states. Since there are no big black holes in supergravity corresponding to these states, there does not appear to be a supergravity realization of these states predicted dyon degeneracy. This raises the following question. *Do the lightlike and timelike states predicted by the dyon degeneracy formula actually exist in the spectrum and if so what is their macroscopic realization?* It is important to address this question to determine the range of applicability of the dyon degeneracy formula.

To start with, let us emphasize that the lightlike or timelike states are not necessarily pathological even though there is no supergravity solution corresponding to them. The simplest example of a lightlike state is the half-BPS purely electric state in the heterotic frame with winding $w$ along a circle and momentum $n$ along the same circle [23][24]. For such a state, $Q_e^2 = 2nw$ is nonzero but since it carries no magnetic charge, both $Q_m^2$ and $Q_e \cdot Q_m$ are zero and hence the discriminant is zero. The supergravity solution is singular but higher derivative corrections generate a horizon with the correct entropy [25][26][27]. We would like to know if similarly there exist quarter-BPS states that are timelike or lightlike in accordance with the predictions of the dyon degeneracy formula and what their supergravity realization is.

In general, it is not easy to extract closed form expression for the degeneracy in this regime when the discriminant is negative or zero. However, we have already encountered a simple example of a timelike state in §5 which nicely illustrates some of the main issues involved. Consider states with $\left(Q_m^2/2,\, Q_e^2/2,\, Q_e \cdot Q_m\right)$ equal to $(-1, -1, N)$. The discriminant is $1 - N^2$ which can be arbitrarily negative. It is easy to determine the degeneracy of this state from a direct expansion of $\Phi_{10}$ to find $d(-1, -1, N) = N$. Do such states exist in the physical spectrum, and if so what is their supergravity realization that can explain the degeneracy?

It is easy to construct such a state from a collection of winding, momentum, KK5, NS5 states in heterotic description. We choose a convenient representative that makes the supergravity analysis in the following section simpler. We consider heterotic string compactified on $T^4 \times S^1 \times \tilde{S}^1$. Let the winding and momentum around the circle $S^1$ be $w$ and $n$ and around the circle $\tilde{S}^1$ be $\tilde{w}$ and $\tilde{n}$. Similarly, $K$ and $W$ are the KK-monopole and NS5-brane charges associated with the circle $S^1$ whereas $\tilde{K}$ and $\tilde{W}$ are the KK-monopole and NS5-brane charges associated with the circle $\tilde{S}^1$. Note that the state with charge $W$ can be thought of as an NS5 brane wrapping along $T^4 \times \tilde{S}^1$ whereas the states with charges $\tilde{W}$ is wrapping along $T^4 \times S^1$. While the state that magnetically dual to $n$ is $K$ in terms of Dirac quantization condition, the state that is S-dual to $n$ is $W$. Similar comment holds for other states. With this notation, we then choose the charges $\Gamma = (Q_e|Q_m) = (n, w; \tilde{n}, \tilde{w}|W, K; \tilde{W}, \tilde{K})$ to be

$$\Gamma = (1, -1; 0, N|0, 0; 1, -1). \tag{46}$$

This state clearly has $(Q_m^2/2, Q_e^2/2, Q_e \cdot Q_m) = (-1, -1, N)$.

Now, since the discriminant of this state is negative, there is no smooth single-centered black hole corresponding to it. What supergravity configuration can then explain the degeneracy $N$ of these states? It turns out that in supergravity, this state is realized not as a single centered solution but rather with two-centered solution [8]. One center is purely electric with charge vector

$$\Gamma_1 = (1, -1; 0, N|0, 0; 0, 0), \tag{47}$$

and the other purely magnetic with charge vector

$$\Gamma_2 = (0, 0; 0, 0|0, 0; 1, -1), \tag{48}$$

both separated by a distance $L$. The distance between two centers is determined in terms of the charges and the asymptotic values of the moduli by the Denef constraint [28][29].The corresponding supergravity solution exists for our charge configuration with a positive, nonzero value for the distance $L$ both for positive and negative $N$ in a large regions of the moduli space but not for all values of the moduli. We discuss the moduli dependence and the lines of marginal stability in the next section.

It is easy to see that such a two-centered solution has the desired degeneracy in agreement with the prediction from the dyon partition function. Each center individually contributes no entropy because for example the electric center by itself has $Q_e^2/2 = -1$ and hence carries no left-moving oscillations and corresponds to the unique ground state. However, because the charges are not mutually local, there is a net angular momentum $j = (N-1)/2$ in the electromagnetic field. For large $N$, the angular momentum multiplet has $2j+1$ or $N$ states in perfect agreement with the dyon degeneracy formula. We thus see that the states with negative discriminant predicted by the dyon degeneracy formula can be realized physically as multi-centered configurations.

## 9. Lines of Marginal Stability

Since $L$ is the separation between the two centers, it must be positive. This can be ensured for a large region of moduli space but not everywhere. The locus in the moduli space where this quantity becomes negative determines the line of marginal stability. Let us take the radii of the two circles $S^1$ and $\tilde{S}^1$ to be $R_1$ and $R_2$ respectively. Let the asymptotic value of the axion dilaton field $S_\infty$ be $\chi + i/g^2$ where $\chi$ is the asymptotic value of the axion field and $g^2$ is the string coupling. If we keep the radii fixed then we find that the line of marginal stability in the upper half $S_\infty$ plane is given by the equation

$$\chi = N \frac{R_1 R_2}{R_1^2 - 1} \frac{1}{g^2}. \tag{49}$$

This equation defines a straight line in the complex $S_\infty$ [30][9]. Note that the slope of the line is proportional to $N$. For fixed $R_1$ and $R_2$, this defines a curve of marginal stability in the complex $S_\infty$ plane. For positive $N$, the desired two-centered solution exists if $\chi + i/g^2$ lies to the left of the line defined by the equation (49). In this region, the distance between the two centers determined by Denef constraint is positive and finite. After crossing the line of marginal stability, the solution ceases to exist because then there is no solution with positive $L$ to the constraint. As one approaches the line of marginal stability from the left, the distance $L$ between the electric and magnetic centers goes to infinity. In other words, the total state with charge vector $\Gamma$ decays into two fragments with charge vectors $\Gamma_1$ and $\Gamma_2$. The mass $M$ of the state with charge $\Gamma$ is given in terms of the central charge by the BPS formula $M = |Z(\Gamma)|$. Note that the central charges depend not only on the charges but also on the moduli. We would now like to know if the state with charge vector $\Gamma$ can decay into two fragments with charges $\Gamma_1$ and $\Gamma_2$. By charge conservation, we have $Z(\Gamma) = Z(\Gamma_1) + Z(\Gamma_2)$. Since the central charges are complex numbers they satisfy the triangle inequality $|Z(\Gamma)| \leq |Z(\Gamma_1)| + |Z(\Gamma_2)|$. The equality applies only when all three complex numbers have the same phase and are thus parallel two-dimensional vectors. At a generic point in the moduli space, they are non-parallel and we have strict inequality. As a result, by energy conservation the state cannot decay into smaller fragments. However at the curve of marginal stability, all three central charges have the same phase and are parallel. Hence the state with charge vector $\Gamma$ can decay into its fragments with charge vectors $\Gamma_1$ and $\Gamma_2$ by a process that is marginally allowed by the energetics and charge conservation.

Similarly, if $N$ is negative, the straight line defined by (49) has negative slope and a solution with positive $L$ exists only to the right of this line. As we have noted, the S-transformation maps the configuration with $N$ positive to $N$ negative. Hence the line with positive slope gets mapped to a line with negative slope and thus the curves of marginal stability move under S-duality. The fact that a two centered solution exists for both signs and with the correct degeneracy is consistent with our prescription for extracting S-duality invariant spectrum proposed in §5. In the wedge between the two lines defined the two lines of marginal stability for $N$ positive and $N$ negative, both states coexist. In other regions, only one or the other state exists.

The simplicity of the line of marginal stability defined by (49) has a simple and beautiful interpretation from the string web picture reviewed in §7. Indeed a string web made out of strands with certain charges exists only if these charges can be carried by a supersymmetric string in six dimensions. If one crosses a line of degeneration in the moduli space, across which a strand with charges, say, $Q_e + Q_m$ shrinks to zero length and is replaced by a strand with charge $Q_e - Q_m$, the quarter-BPS state will decay if no supersymmetric string with charge $Q_e - Q_m$ exists. The line of degeneration is simply the line at which a string of charge $Q_e$ along one cycle of the torus and a string of charge $Q_m$ along the other can be simultaneously supersymmetric. This is equivalent to the requirement that the phase of $S$ is the same as the angle between the central charge vectors for $Q_e$ and $Q_m$, that defines a straight line in the $S$ plane. In the present case $Q_e = (1,-1,0,N)$ and $Q_m = (0,0,1,-1)$, hence $Q_e \pm Q_m = (1,-1,\pm 1, N \mp 1)$. $\frac{1}{2}(Q_e \pm Q_m)^2 = -1 \pm N$, but a BPS string with charge $Q$ must have $Q^2/2 \geq -1$. Hence the line of degeneration of the string web is indeed a line of marginal stability.

It is not surprising that the existence of quarter-BPS dyons depends on the moduli and that there are lines of marginal stability which separate the regions where the state exists from where it does not exist. This phenomenon is well known in the field theory context [31].

The lines of marginal stability are also consistent with the choices for the contours for extracting the degeneracies used in §5. For example, for states with $N > 0$, the two-centered solution exists to the left of the line of marginal stability with degeneracy $N$. In this region one must expand the partition function around $y = 0$ as in (26) resulting in $d(-1,-1,N) = N$ consistent with supergravity realization. On the other hand, the the right of the line of marginal stability (49) the two-centered solution does not exist and the state decays hence the degeneracy is zero. Corresponding to this region of the moduli space one must expand the partition function around $y^{-1} = 0$ as in (29) resulting in $d(-1,-1,N) = 0$, again consistent with supergravity realization.

## 10. Irreducibility Criteria

The appearance of genus two Riemann surfaces and the reasoning above in terms of string webs raises the question of possible contributions of higher genus Riemann surfaces. We would like to therefore like to address the question of when the genus-two Siegel modular form completely captures the degeneracies.

There are various derivations of the dyon degeneracy formula, but often they compute the degeneracies for a specific subset of charges, and then use duality invariance to extend the result to generic charges. Such an application of duality invariance assumes in particular that under the duality group $SO(22,6,\mathbf{Z})$ the only invariants built out of charges would be $Q_e^2$, $Q_m^2$, and $Q_e \cdot Q_m$. This assumption is incorrect. If two charges are in the same orbit of the duality group, then obviously they have the same value for these three invariants. However the converse is not true. In general, for arithmetic groups, there can be discrete invariants which cannot be written as invariants of the continuous group.

An example of a non-trivial invariant that can be built out of two integral charge vectors is $I = gcd(Q_e \wedge Q_m)$, *i.e.*, the gcd of all bilinears $Q_e^i Q_m^j - Q_e^j Q_m^i$. Our goal is to show that the genus-two dyon partition function correctly captures the degeneracies if $I = 1$. Note that half-BPS states have $I = 0$ and hence are naturally associated with a genus-one surface. If $I > 1$, then there are additional zero modes for the dyon under consideration and it would be necessary to correctly take them into account for counting the dyons.

To understand the relation between the value of $I$ and the possible variety of string webs that may describe a dyon with given charges, it is useful to consider a graph in the space of charges that is topologically dual to the string web. A dual graph is constructed as follows. For every face of the web associate a vertex in the dual graph. If two faces $A$ and $B$ in the web share an edge then the corresponding vertices $A'$ and $B'$ in the dual graph are connected by a vector that is equal in magnitude to the central charge of the edge but rotated by $\pi/2$ in orientation compared to the edge. Recall that each edge in the string web carries a central charge and that the relative angles between the edges mimic the angles between the corresponding central charge vectors. A junction has three faces and three edges which maps to a triangle in the dual graph with three vertices and three edges. Charge conservation at each junction means that the vector sum of the three edge vectors is zero. This then guarantees that the sides of the dual triangle actually close, as their vector sum is zero. A string web constructed from a period array of junctions then corresponds to a triangulation in the dual graph.

Now, the vertices of the dual graph will sit at integral points of the charge lattice, on the plane defined by the vectors $Q_e$ and $Q_m$. The graph will have a fundamental cell with sides $Q_e$ and $Q_m$. Our invariant $I$ counts the number of integral points inside the fundamental cell. In this dual description, it is clear geometrically that $Q_e^i Q_m^j - Q_e^j Q_m^i$ are the various components of the area 2-form associated with the fundamental cell. If all the components do not have common factor then the fundamental parallelogram does not have any integral points either on the edges or inside. Hence the original web is "irreducible" in that it has not internal faces. The corresponding M5-brane worldsheet has then genus two. For further details see [30].

To summarize, the genus-two partition function captures the degeneracies of a dyon only if its electric and magnetic charges satisfy the irreducibility criterion

$$gcd(Q_e^i Q_m^j - Q_e^j Q_m^i) = 1. \tag{50}$$

## 11. Conclusion

We have seen that the dyon partition function has a rich structure and contains nontrivial information about the nonperturbative spectrum of quarter-BPS dyons. For large charges, the degeneracies predicted by the partition function are in impressive agreement with the entropy of corresponding dyonic black holes to leading and to subleading order. The entropy of black holes thus supplies us with very quantitative useful information about the fundamental degrees of freedom and structure of quantum gravity. For small charges, the partition function reveals in intricate structure of lines of marginal stability and multi-centered configuration. Some of these physical properties of the dyon spectrum need to be investigated further.

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