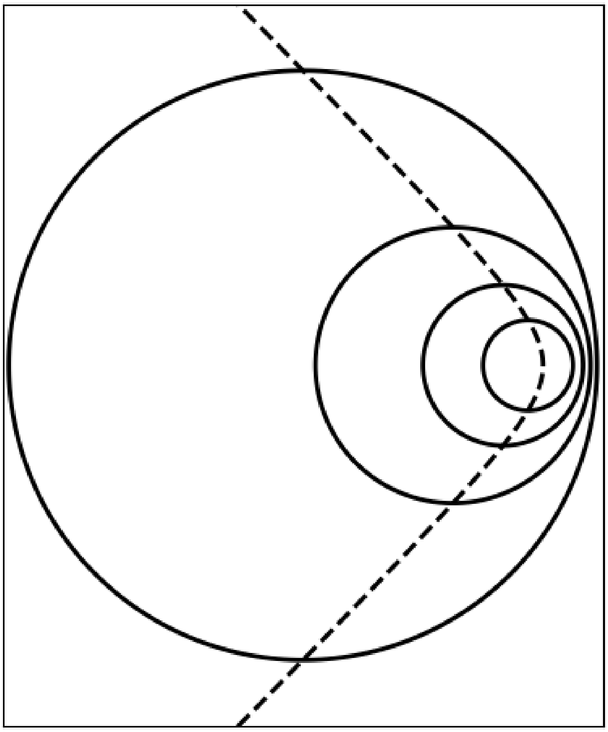
ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

Lecture notes on: Physics and Dynamics of the Ocean

Riccardo Farneti



EARTH SYSTEM PHYSICS DIPLOMA PROGRAMME
2025-2026

Preface

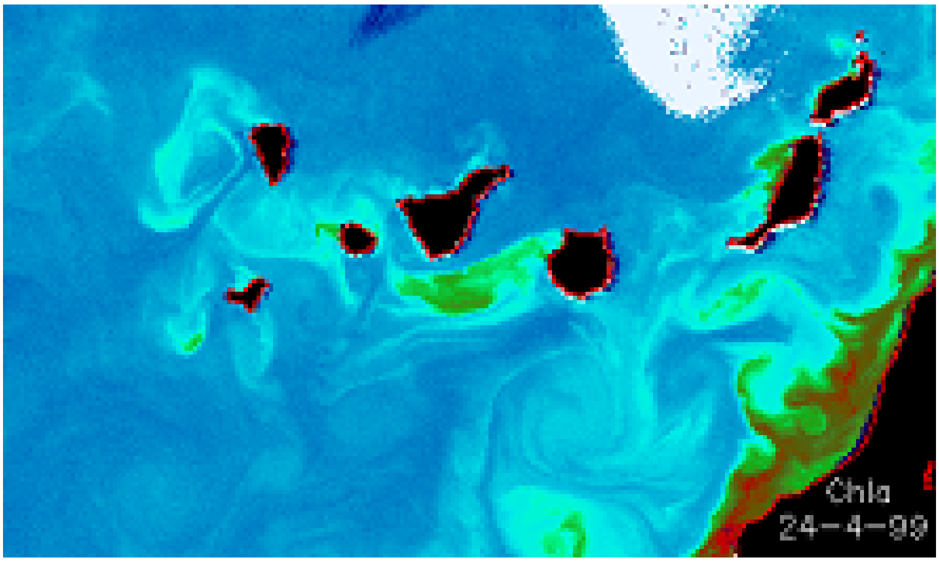
These notes form the basis for the *Physics and Dynamics of the Ocean* Course of the Abdus Salam International Centre for Theoretical Physics Diploma Programme in Earth System Physics.

Some of the material on these notes is *borrowed* from *Vallis* (2006); *Talley et al.* (2011); *Olbers et al.* (2012); *Vallis* (2019). Students are encouraged to supplement these notes with readings of those and more textbooks.

The notes will grow in time, and comments and suggestions are highly appreciated.

This is the version from February 11, 2026.

Riccardo Farneti
(rfarneti@ictp.it)



Satellite image from SeaWIFS showing surface Chl-a concentration on the north-west coast of Africa around the Canary Islands. Notice the downstream cyclonic and anticyclonic eddies formed by the interaction of the Canary Current with the islands and the coastal upwelling shedding filaments and eddies.

Contents

Chapter 1

Introduction

We shall discuss the basic principles and dynamics setting up the large-scale ocean circulation. We will make use of all concepts we have developed in our previous courses (Geophysical Fluid Dynamics, Physics of the Ocean) and we will probably see some topics that have been mentioned in other courses (e.g., Atmospheric dynamics). But, *repetita iuvant!*. And those same principles will be applied here to ocean dynamics, perhaps in a revised way.

The large-scale ocean circulation can be broadly divided into two different kinds, a *horizontal surface wind-driven circulation* and a *meridional deep buoyancy- and wind-driven circulation*, although the distinction is only a bad approximation as they are intimately connected. This is particularly so for the meridional overturning which is driven both by buoyancy and wind at the surface.

Ocean circulation theory is based on the very same principles that drive atmospheric circulation, and many theories have been borrowed from the meteorological and atmospheric fields. Of course, the ocean is just another geophysical fluid, and as such it is governed by all GFD conservation principles, forces and instabilities you have been exposed already. There are two main differences with respect to the atmosphere that are worth pointing out now.

- First, contrary to the atmosphere, the ocean is heated and cooled from above.
- Second, ocean circulation is often constrained by the presence of continents, and this will alter the structure and dynamics of the flow.

These two peculiarities will explain some of the differences between oceanic

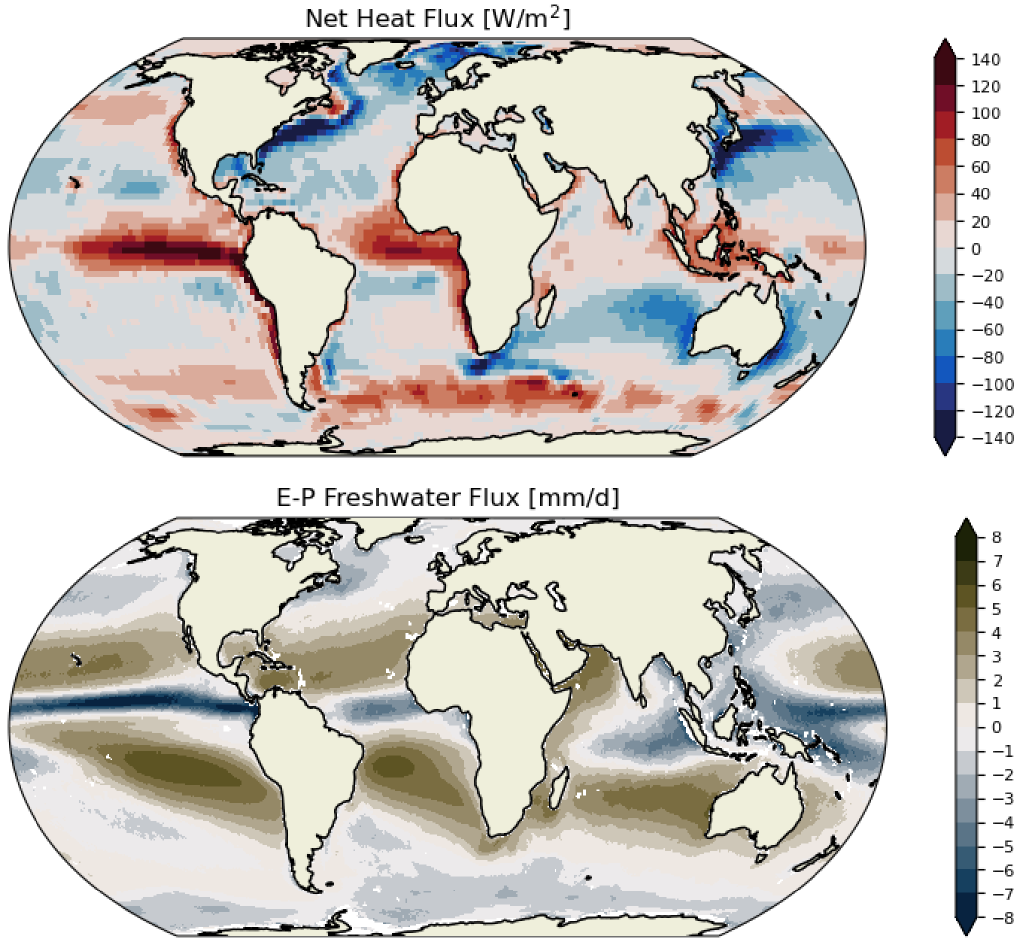


Figure 1.1: Net heat and freshwater fluxes computed from the NCEP/NCAR reanalysis for the period 2010-2019 *Kalnay et al.* (1996).

and atmospheric circulation, as well as some geographical uniqueness in ocean circulation.

The ocean is largely driven by surface wind stress (Fig. 3.7) –actually not the stress but its curl! ... see later–, and common patterns arise in the surface wind-driven large-scale circulation of all different basins (Fig. 1.3). They all have just a few common ingredients, and these will qualitatively explain the main features of the wind-driven gyres. The Southern Ocean is a rather different story, and it will be discussed separately.

However, buoyancy forcing –the sum of surface heat and freshwater fluxes– and the latitudinal extent of the ocean basins, will alter the way the surface of each basin is buoyancy forced, with profound implications for

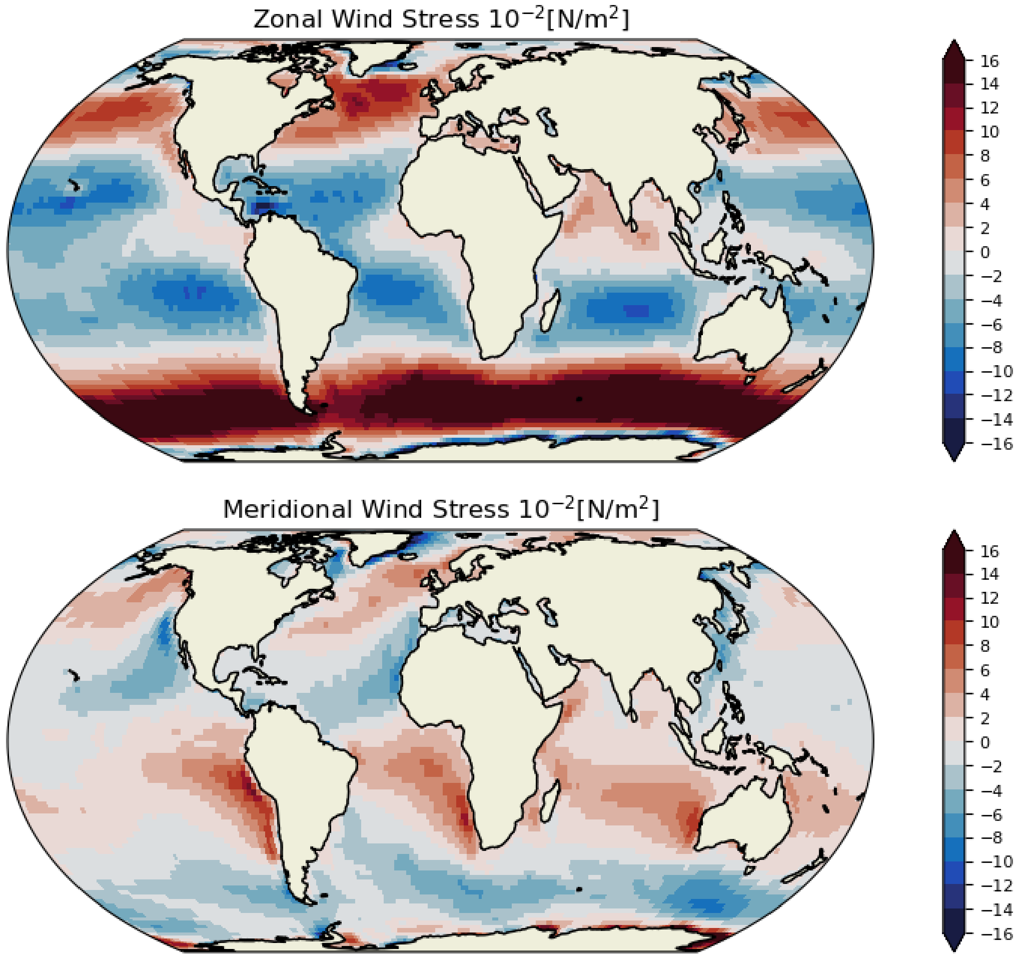


Figure 1.2: Surface zonal and meridional components of the wind stress computed from the NCEP/NCAR reanalysis for the period 2010-2019 *Kalnay et al.* (1996).

the interior temperature and salinity structure as well as deep circulations of the oceans. Once we have highlighted the major circulations and their relations, a clear picture of the interior and meridional circulation will also appear.

There is a good observational and theoretical understanding of the major processes responsible for the Meridional Overturning Circulation (MOC) (Fig. 1.4). This is not the same in each and every basin (Fig. 1.5), and fundamental differences exist giving rise to shallow and deep circulations, responsible for different degrees of meridional energy and mass transports. Our present models capture these features reasonably well

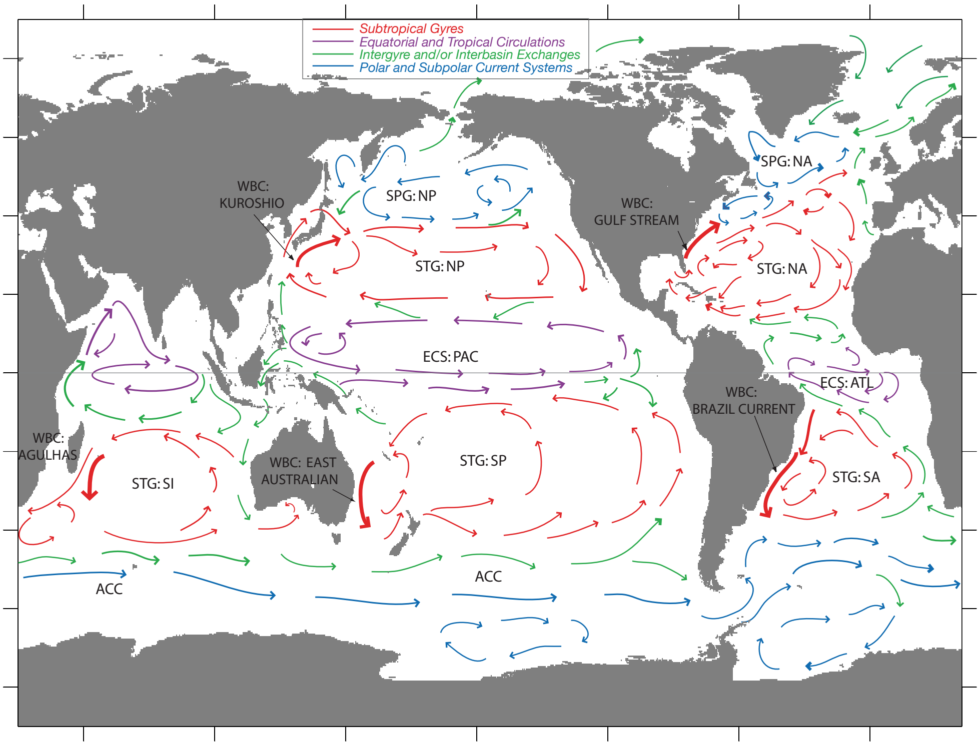


Figure 1.3: A schema of the main currents of the global ocean [from *Vallis* (2006)].

(Fig. 1.6), although many small-scale effects are missing or poorly parameterized, and most importantly the variability of this circulation is not well understood (let alone its future evolution!).

Observations indicate that approximately 90% of extra heat of anthropogenic origin stored in the climate system is taken up by the ocean. As of today, ocean warming has already led to significant consequences on ocean circulation, altering ocean biogeochemistry, rising sea level, impacting atmospheric dynamics, melting sea ice and ice sheets. However, a comprehensive understading of the dynamics, time and spatial scales of heat redistribution and addition into the ocean is still lacking. Also, feedbacks between changes in ocean heat content and altered ocean dynamics on the stability of ice sheets, and ultimately of large-scale ocean currents regulating our climate, are still to be fully explored both theoretically and with the aid of climate models. Reducing this knowledge gap and uncertainties is fundamental for understading and improving future climate projections.

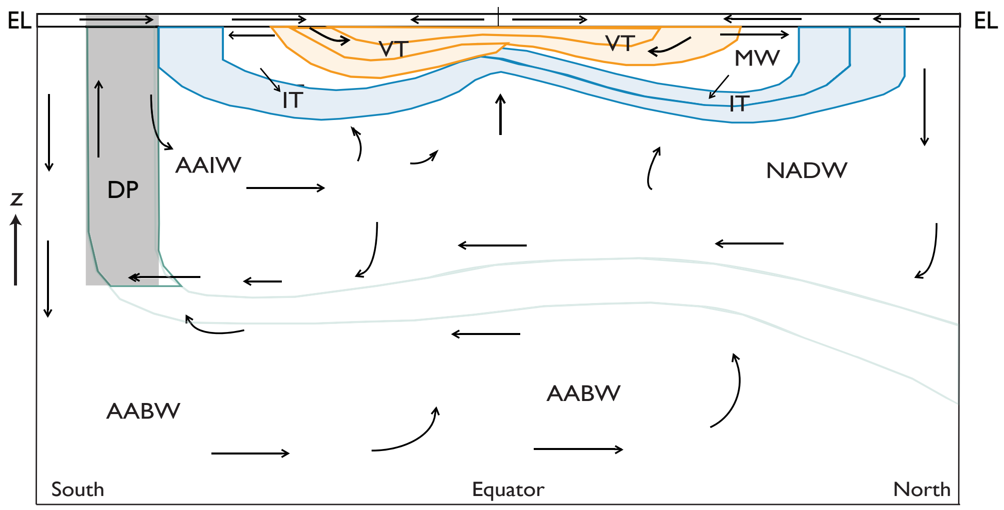


Figure 1.4: A schema of the stratification and overturning circulation. [from *Vallis* (2006)]

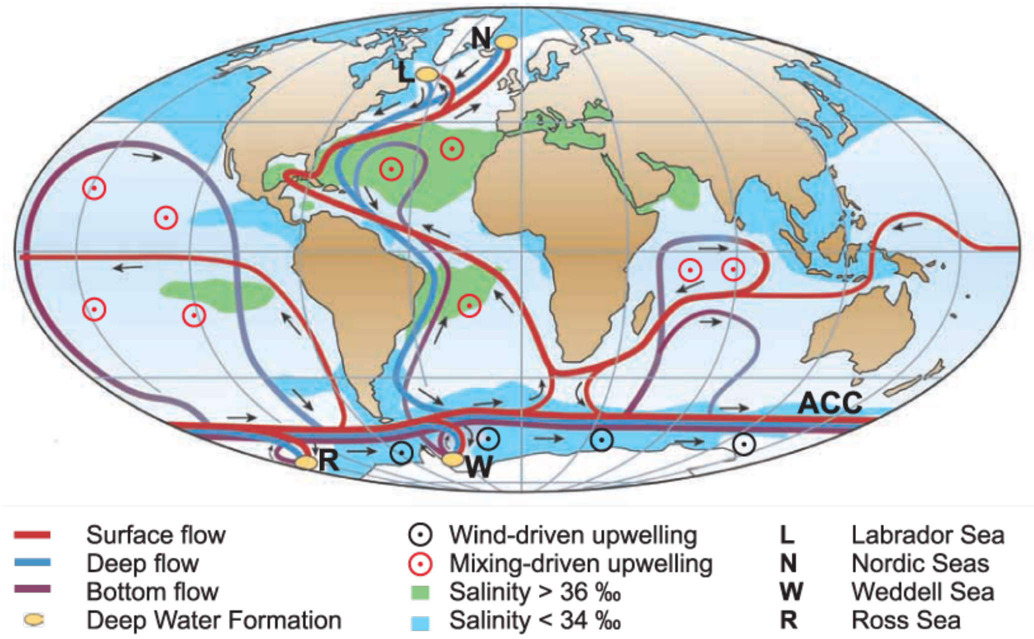


Figure 1.5: A schema of the thermohaline circulation (THC), or the Meridional Overturning Circulation (MOC). [from *Kuhlbrodt et al.* (2007)]

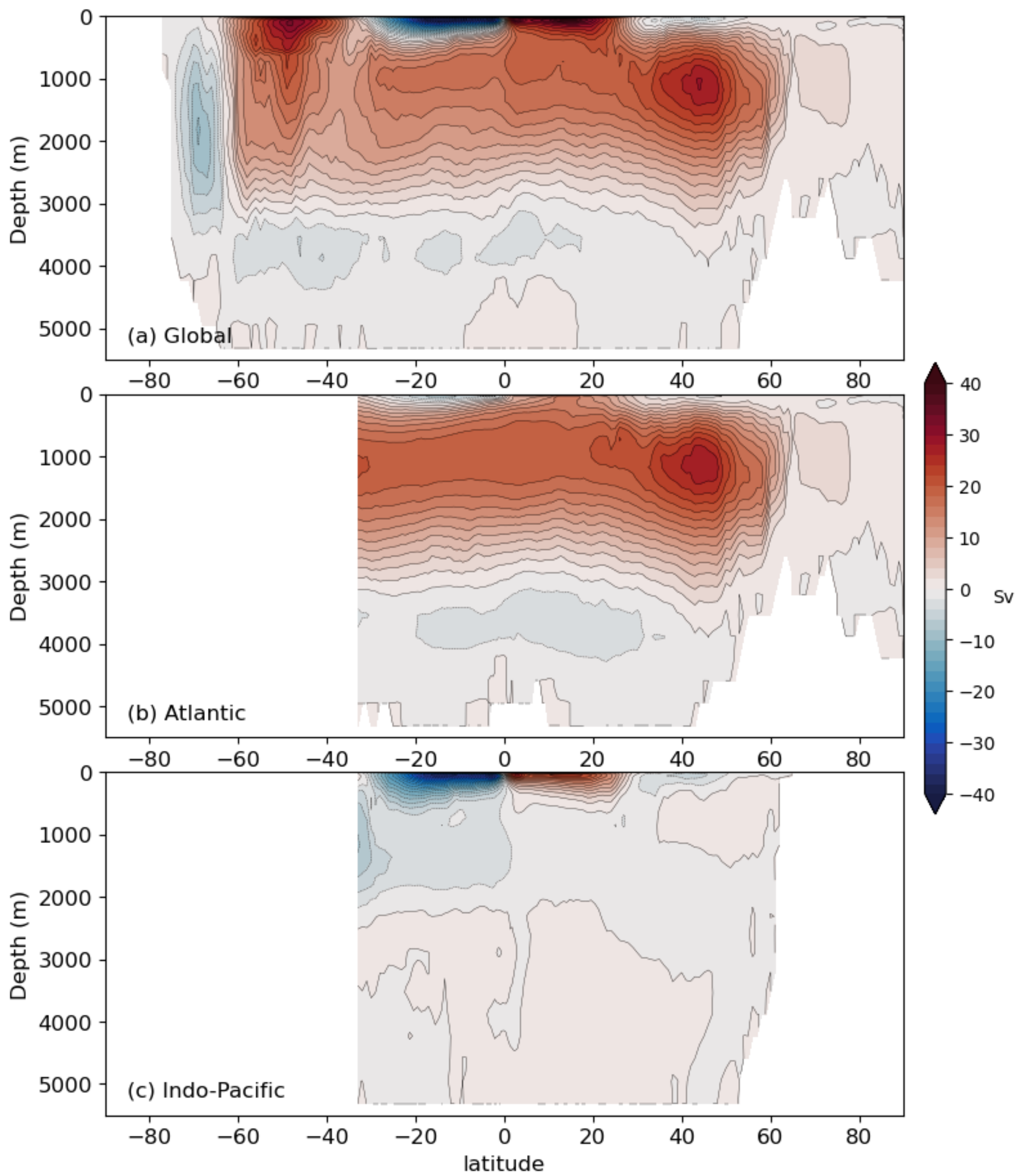


Figure 1.6: The global MOC as computed from a Coupled General Circulation Model (CGCM). We clearly see the presence of the North Atlantic Deep Water cell, the interhemispheric meridional circulation, a locally-circulating deacon Cell, and two SubTropical Cells. Each meridional cell is driven by different dynamics and all together set up the global ocean circulation.

Chapter 2

Thermodynamics of Seawater

2.1 Temperature, Salinity and Density in the Ocean

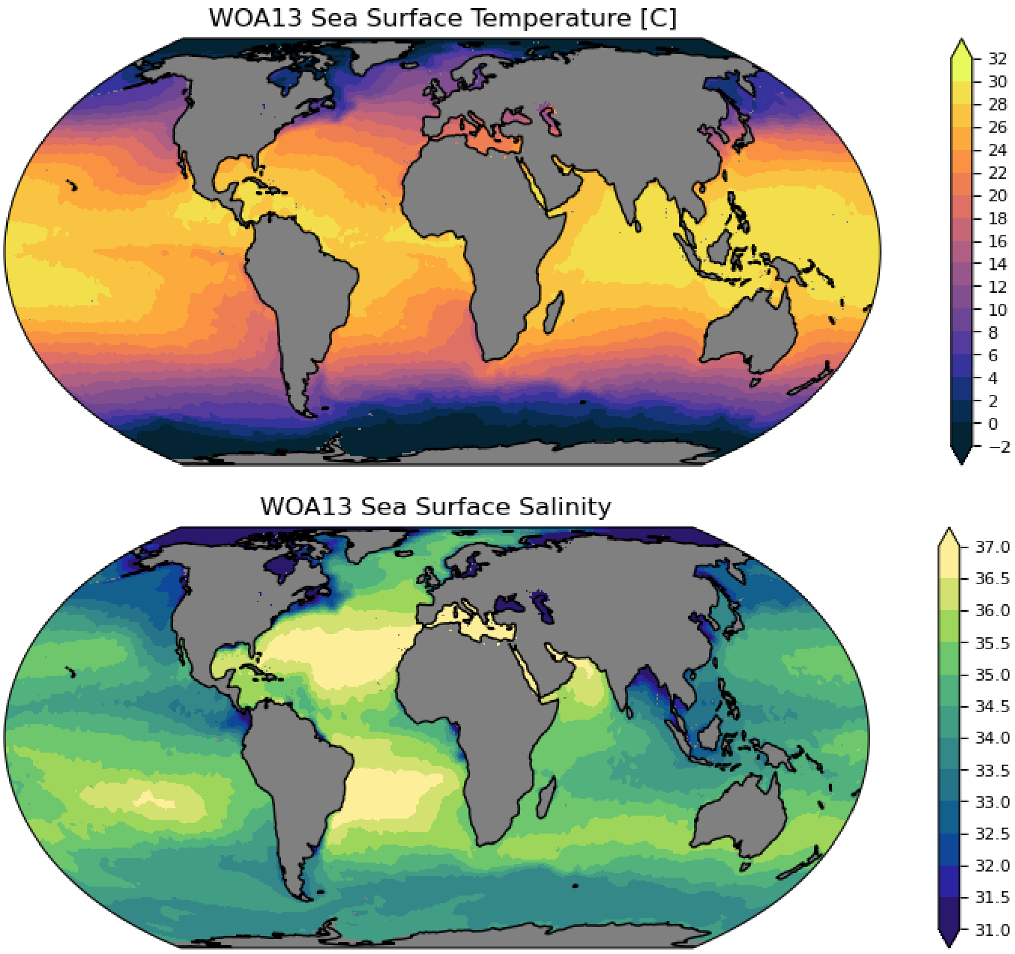


Figure 2.1: *Climatological mean (2005-2012) Sea Surface Temperature and Sea Surface Salinity for the global ocean from in situ profile data (World Ocean Atlas 2013 version 2) at 0.25 degree horizontal resolution.*

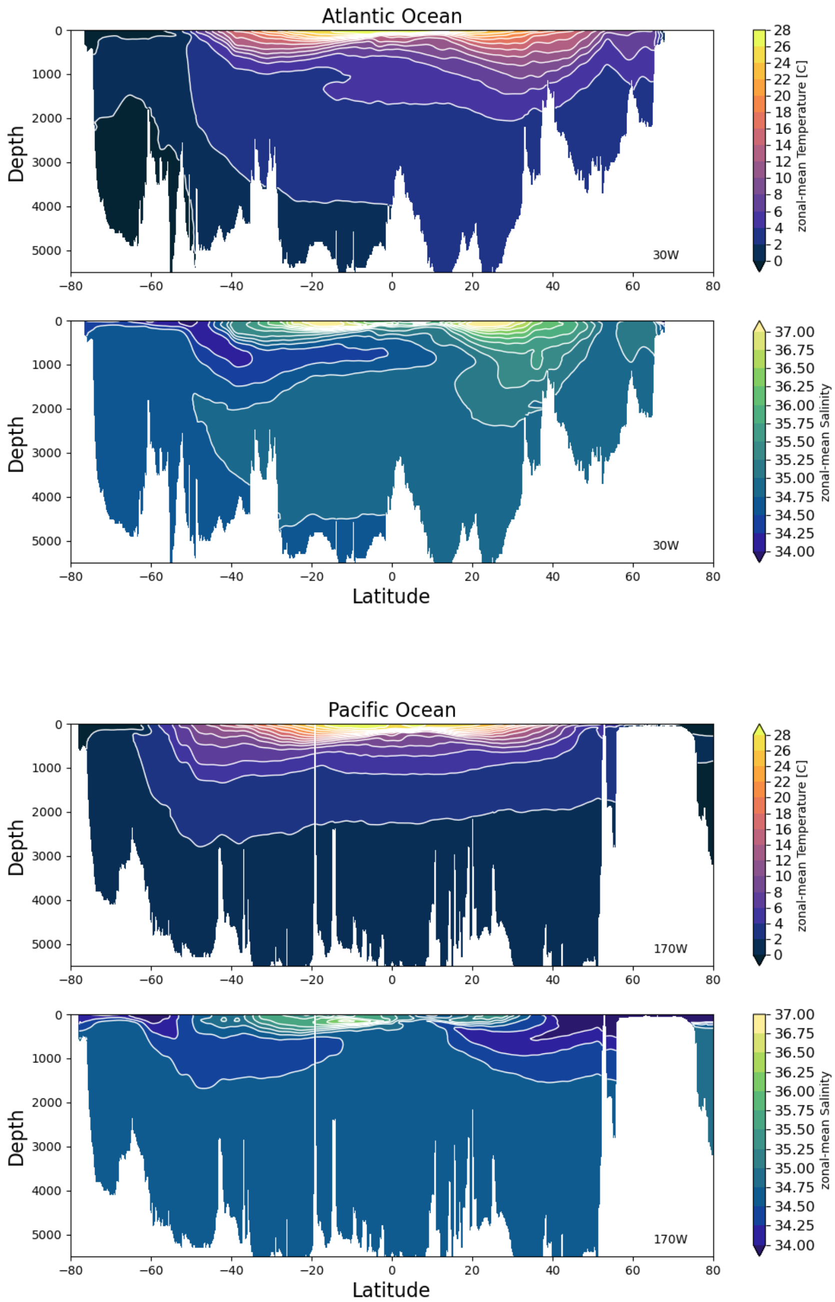


Figure 2.2: *Zonal-mean of the climatological (2005-2012) Temperature and Salinity for the Atlantic Ocean at 30W and Pacific Ocean at 170W from in situ profile data (World Ocean Atlas 2013 version 2) at 0.25 degree horizontal resolution.*

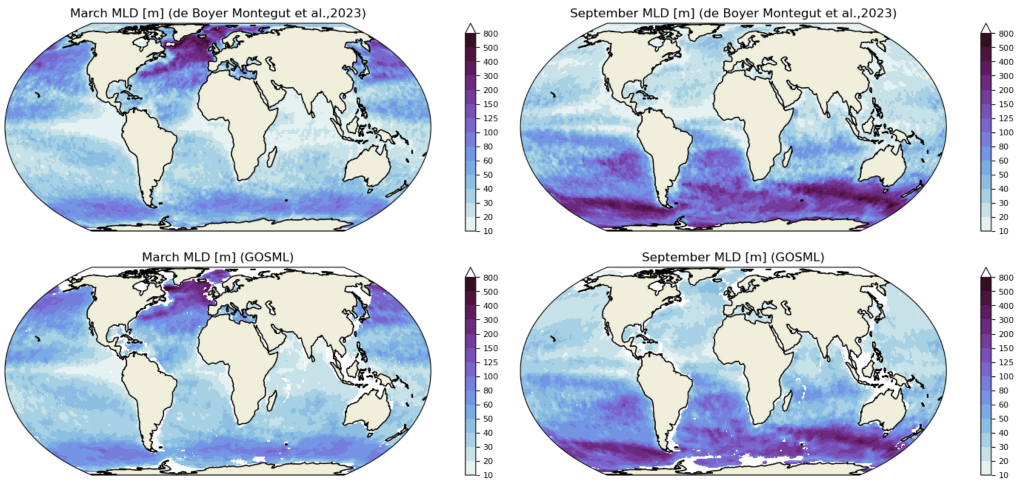


Figure 2.3: *Mixed layer Depth (in m) computed from two different datasets (de Boyer Montegut, 2023 and GOSML) for March and September.*

Chapter 3

Air-Sea interactions

3.1 Air-sea exchange of heat

About half the solar energy reaching Earth is absorbed by the ocean and land, where it is temporarily stored near the surface. Only about a fifth of the available solar energy is directly absorbed by the atmosphere. Of the energy absorbed by the ocean, most is released locally to the atmosphere, mostly by evaporation and infrared radiation. The remainder is transported by currents to other areas especially mid latitudes. Note that heat is the amount of thermal energy transferred from one body to another because of the temperature difference between those bodies.

Heat lost by the tropical ocean is the major source of energy needed to drive the atmospheric circulation. And, solar energy stored in the ocean from summer to winter helps ameliorate Earth's climate. The thermal energy transported by ocean currents is not steady, and significant changes in the transport, particularly in the Atlantic, may have been important for the development of the ice ages. For these reasons, oceanic heat budgets and transports are important for understanding Earth's climate and its short and long term variability.

Changes in energy stored in the upper layers of the ocean result from a local imbalance between input and output of heat through the sea surface. This transfer of heat through the surface is called a heat flux. The flux of heat and water also changes the density of surface waters, and hence their buoyancy. As a result, the sum of the heat and water fluxes is often called the buoyancy flux.

The flux of energy to deeper layers is usually much smaller than the flux through the surface. And, the total flux of energy into and out of the

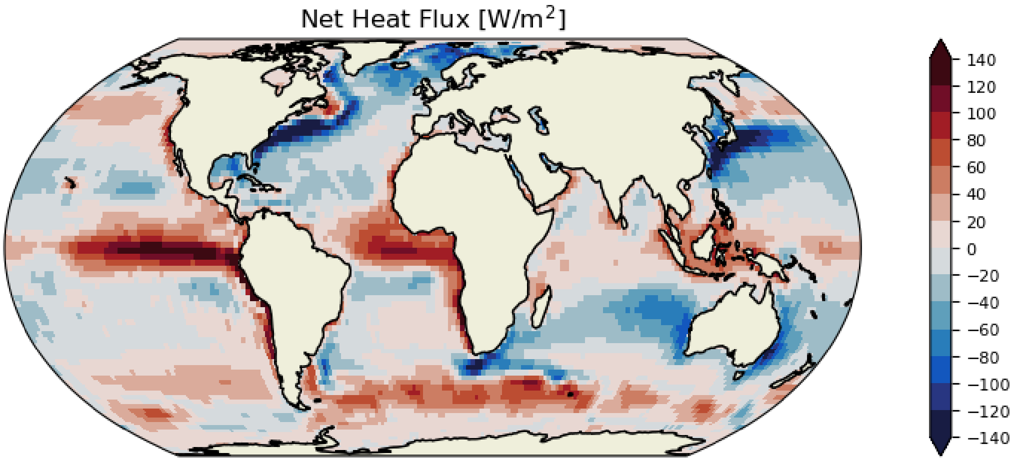


Figure 3.1: *Long-term means of surface net heat flux. Positive values indicate a flux into the ocean. Data from the NCEP/NCAR reanalysis (Kalnay et al., 1996).2019 only!*

global ocean must be zero, otherwise the ocean as a whole would heat up or cool down. The sum of the heat fluxes into or out of a volume of water is the heat budget.

3.1.1 Heat budget at the surface

The major terms in the budget at the sea surface are:

1. **insolation**, Q_{sw}, the flux of sunlight into the sea. Represents the radiative heat flux from the incoming solar radiation minus that reflected. The average value of incoming solar radiation at the top of Earth's atmosphere is 342 W m^{-2}, although this value varies considerably with latitude and season. Some of the solar radiation is absorbed by the atmosphere or reflected back to space by clouds and aerosols, never reaching the ocean. Furthermore, the ocean doesn't absorb all of the shortwave radiation that reaches its surface; some is also reflected back to space. The albedo α describes how much radiation is absorbed versus reflected. Therefore, the shortwave heating is given by
$$Q_{sw} = (1 - \alpha) Q_{sw}^{inc}. \tag{3.1}$$
The net solar heat input at the sea surface ranges 250 W/m^2 in the tropics to 50 W/m^2 at high latitudes. This differential solar heating over the globe is the powerhouse of the atmosphere and ocean.

2. **Net Infrared Radiation**, $\mathcal{Q}_{lw}$, net flux of infrared radiation from the sea. It is the radiative heat flux over the range of wavelengths emitted from the sea surface, dominated by infrared radiation, so it is negative. Because of its much lower temperature, the earth emits radiation in the longwave (i.e. infrared) band. On a global scale, emission is the main way the planet balances the incoming solar radiation to maintain its thermal equilibrium. The power of emitted radiation is strongly dependent on temperature; specifically, the Stefan-Boltzman law states that

$$\mathcal{Q}_{lw} = -\sigma T^4. \tag{3.2}$$

The fact that $\mathcal{Q}_{lw}$ is negative definite means that longwave emissions acts to cool the ocean.

3. **Sensible Heat Flux**, $\mathcal{Q}_{sh}$, the flux of heat out of the sea due to conduction. Turbulent transfer of heat across the sea surface as a function of the air-sea temperature difference. Radiative fluxes transmit heat energy over long distances, e.g.from the sun to the earth. Sensible heat exchange instead depends on direct molecular contact between air and water. The molecules of the two fluids bump against each other, thereby exchanging heat energy. When the ocean and air temperature are the same, the sensible heat flux is therefore zero. When they differ, heat is exchanged in such as way as to homogenize the temperature. This is analogous to the phenomenon of "Newtonian cooling" commonly studied in introductory physics courses.

The rate of exchange is highly dependent on the sea state, and in particular on the winds, which generate breaking waves and turbulence. It is easy to imagine how breaking waves enhance the exchange of heat: they literally pull air down into the water (bubbles) and splash water up into the air (sea spray). It is much harder to quantify this process mathematically. Nevertheless, laboratory experiments and field campaigns have permitted us to develop empirical formulas to do so:

$$\mathcal{Q}_{sh} = \rho^a C_p^a C_H |\mathbf{u_{10}^a} - \mathbf{u^o}|(T_{10}^a - T^o) \tag{3.3}$$

All of the complexity and difficulty of turbulent air-sea exchange is absorbed into the parameter C_H.

The annual mean contribution of $\mathcal{Q}_{sh}$ to $\mathcal{Q}$ is almost negligible. This is because, on average, the T and T_{10}^a are very close. There is, however, a strong seasonal cycle in sensible heat exchange. Because of the large ocean heat capacity, the ocean is usually colder than the air in summer and warmer than the air in the winter. This drives important seasonal changes in circulation

4. **Latent Heat Flux**, $\mathcal{Q}_{lh}$, the flux of heat carried by evaporated water. Turbulent transfer of evaporated water, and heat is used to enable the phase change from liquid to vapour. This process requires thermodynamic energy and therefore extracts heat from the ocean. The latent exchange nearly always dominates over the sensible exchange.

$$\mathcal{Q}_{lh} = -L_e E, \tag{3.4}$$

   where E is the evaporation rate and L_e is the latent heat of vaporization. The latent heat flux always cools the ocean

5. **Advection** $\mathcal{Q}_{adv}$, heat transported by ocean currents.

Conservation of heat requires:

$$\mathcal{Q} = \mathcal{Q}_{sw} + \mathcal{Q}_{lw} + \mathcal{Q}_{sens} + \mathcal{Q}_{latent} + \mathcal{Q}_{adv} \tag{3.5}$$

where Q is the resultant heat gain or loss. Units for heat fluxes are W/m^2. The product of flux times surface area times time is energy in joules.

There is no local heat balance, instead there is a net heating gain over the tropical regions and a localised loss of heat at high latitudes. To keep the ocean in a steady state, ocean circulation must therefore transport heat from equator to pole ($\mathcal{Q}_{adv}$).

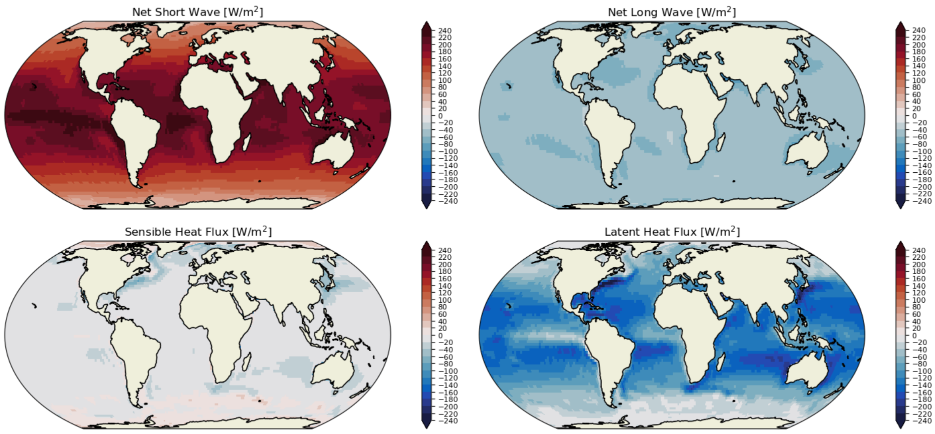


Figure 3.2: *Long-term means of net surface shortwave, net surface longwave radiation, surface sensible and latent heat fluxes. Positive values indicate a flux into the ocean. Data from the NCEP/NCAR reanalysis (Kalnay et al., 1996).2019 only!*

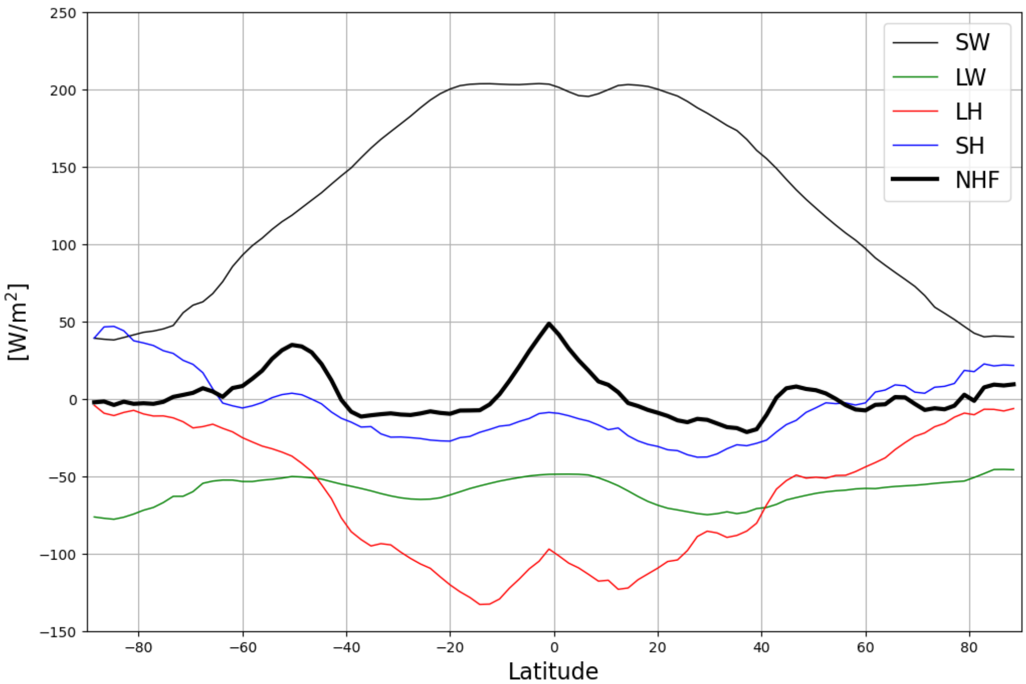


Figure 3.3: *The zonal-mean of net surface shortwave and net surface longwave radiation, surface sensible and latent heat fluxes, and the net heat flux. Positive values indicate a flux into the ocean. Data from the NCEP/NCAR reanalysis (Kalnay et al., 1996).2019 only!*

3.1.2 The mixed layer

The term 'mixed' refers to a given physical parameter of the ocean state (e.g. temperature, density, ...) that is assumed to be mixed and homogeneous to a certain level (e.g. regarding some space/time scales), from the surface down to the considered mixed layer depth (MLD). Here we want to estimate mixed layer over at least a daily cycle, and no more than a few days. This represents the depth over which surface fluxes have been recently mixed and integrated and is a characteristic timescale of air-sea interactions. The mixed layer depth is a thus a Density-Mixed Layer Depth, or Isopycnal Layer Depth.

The surface MLDs are estimated directly on individual profiles with data at observed levels. MLD is defined through the threshold method with a finite difference criterion from a near-surface reference value. A linear interpolation between levels is then used to estimate the exact depth at which the difference criterion is reached. The reference depth is set at 10 m to avoid a large part of the strong diurnal cycle in the top few meters of the ocean. The fixed criterion in density is 0.03 kg/m^3 difference from surface:

$$\text{MLD} = \text{depth where}(\sigma_0 = \sigma_0(10m) + 0.03 \quad \text{kg m}^{-3}). \tag{3.6}$$

See de Boyer Montegut et al. JGR 2004 for further details about the choice of the criterion).

Mixed layer heat budget equation

In general, two physical processes can cause $\mathcal{H}$ to vary over time within a surface layer of the ocean: an exchange of heat with the atmosphere and advection of heat within the ocean:

$$\frac{\partial \mathcal{H}}{\partial t} = \mathcal{Q} - \mathbf{u} \cdot \nabla \mathcal{H}, \tag{3.7}$$

where $\mathcal{Q} = \mathcal{Q}_{sw} + \mathcal{Q}_{lw} + \mathcal{Q}_{sens} + \mathcal{Q}_{latent}$

The heat content, per unit area, is

$$\mathcal{H} = h\rho C_p T. \tag{3.8}$$

Now considering $\rho \sim \rho_0$ within the mixed layer and h to be constant, these heat fluxes drive a temperature change over a surface mixed layer given by

$$\boxed{\frac{DT}{Dt} = \frac{\mathcal{Q}}{\rho_0 C_p h} - \mathbf{u} \cdot \nabla T} \tag{3.9}$$

Annual gain in heat in the tropics and loss at high latitudes is offset by an ocean heat transport, generally directed poleward, $\mathcal{Q}_{adv}$. Consider the case in which $\mathcal{Q} = 0$, so that temperature advection is the only remaining process causing a change in T in the mixed layer

$$\frac{DT}{Dt} = -\mathbf{u} \cdot \nabla T = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z}. \tag{3.10}$$

So far we have considered h to be a constant, but the thickness of the mixed layer varies in time, mostly through heating and cooling at the surface and turbulent wind mixing. Using Fick's laws of diffusion

$$F_x = -A\frac{\partial C}{\partial x}, \tag{3.11}$$

where F_x is the flux in the x direction, A is a diffusivity coefficient. Also, the time rate of change of a concentration due to diffusive fluxes will be proportional to the second spatial derivative of the concentration

$$\frac{\partial C}{\partial t} = A\frac{\partial^2 C}{\partial x^2}, \tag{3.12}$$

or

$$\frac{\partial C}{\partial t} = A\nabla^2 C. \tag{3.13}$$

This is our starting point for the parameterization of ocean mixing, where C is any physical property and A is the efficiency of turbulent eddies. A local temperature change due to turbulent mixing by eddies is

$$\boxed{\frac{\partial T}{\partial t} = K\nabla^2 T}. \tag{3.14}$$

What is the value of $K = (K_x, K_y, K_z)$? Difficult question, but $K_h >> K_z$ reflecting the fact that the ocean is stably stratified in the vertical, and mixing along isopycnals requires less work than diapycnal mixing. The contribution to the ocean mixed layer heat budget by turbulent mixing is thus

$$\boxed{\frac{\partial T}{\partial t} = K_h \nabla_h^2 T + K_z \frac{\partial^2 T}{\partial z^2}}. \tag{3.15}$$

Turbulent mixing is important in the upper ocean and can significantly modify the surface temperature. When mixing occurs, for example because of strong winds reaching the depth h, water from below the mixed layer can be entrained into the mixed layer, resulting in deepening of the mixed layer and a change in its temperature. The vertical velocity, called entrainment velocity w_e is

$$w_e = \frac{\partial h}{\partial t} + w_{z=h}, \tag{3.16}$$

representing both deepening of the mixed layer and an upwelling velocity. Hence, both entrainment processes are important in the vertical mixing of temperature. Also, neglecting horizontal components

$$\frac{DT}{Dt} = -w\frac{\partial T}{\partial z} = -w_e\frac{\Delta T}{h}, \tag{3.17}$$

and as the mixed layer gets shallower, small deepening or modest upwelling will result in large temperature tendencies.

The total heat budget for a mixed layer is

$$\boxed{\frac{DT}{Dt} = \frac{\mathcal{Q}}{\rho_0 C_p h} - \mathbf{u}\cdot\nabla T + K_h\nabla_h^2 T + K_z\frac{\partial^2 T}{\partial z^2}} \tag{3.18}$$

(a) The change in temperature ΔT of the water is related to change in energy ΔE through:

$$\Delta E = C_p m \Delta T \tag{3.19}$$

where m is the mass of water being warmed or cooled, and C_p is the specific heat of sea water at constant pressure, $C_p \sim 4.0\times 10^3$ J kg^{-1} K^{-1}. Thus, 4,000 Joules of energy are required to heat 1.0 kg of sea water by 1.0°K.
Estimate how many Joules are required to heat 1.0 kg of air by 1.0°K.

(b) Estimate the thickness of the ocean that holds as much heat as the overlying atmosphere, where the amount of heat $\mathcal{H}$ required to raise the temperature of the atmosphere or ocean by ΔT is given by

$$\mathcal{H} = \rho C_p A D \Delta T, \tag{3.20}$$

where ρ is density, C_p is heat capacity, A is horizontal area and D is the vertical scale. Assume $\rho \sim 1$ Kg m^{-3} for the atmosphere and 10^3 for the ocean, $C_p \sim 1000$ J kg^{-1} K^{-1} for the atmosphere and 4000 for the ocean, D of 10 km for the atmosphere, $\Delta T = 1$ K and $A = 1\text{m}^2$.

3.2 Air-sea freshwater flux and surface salinity

The mass of salt M_s in a volume of seawater, per unit area, is

$$M_s = \rho h S, \tag{3.21}$$

and tendencies in the mass will be expressed as

$$\frac{\partial M_s}{\partial t} = \rho h \frac{\partial S}{\partial t}. \tag{3.22}$$

But the salt itself is not exchanged with the atmosphere, only freshwater, so a mass flux of salt is not very practical nor useful.

Fresh water is exchanged between the atmosphere and ocean mainly via precipitation P and evaporation E, but also from river runoff R. Units for the freshwater flux are m y^{-1} or mm d^{-1}, from the volume flux of fresh water ($\mathrm{m}^3\ \mathrm{y}^{-1}$) exchanged per unit area (m^2). Transfer of fresh water from the ocean to the atmosphere will increase the concentration of salts in the surface mixed layer of the ocean.

We define a virtual mass flux of salt as $(E - P)S$

$$\frac{\mathrm{D}\, S}{\mathrm{D}\, t} = \frac{(E-P)}{\rho_0 h} S - \mathbf{u} \cdot \nabla S. \tag{3.23}$$

Similarly to the heat budget, we can write

$$\boxed{\frac{DS}{Dt} = \frac{(E-P)}{\rho_0 h} S - u\frac{\partial S}{\partial x} - v\frac{\partial S}{\partial y} - w_e \frac{\Delta S}{h} + K_h \nabla_h^2 S + K_z \frac{\partial^2 S}{\partial z^2}} \tag{3.24}$$

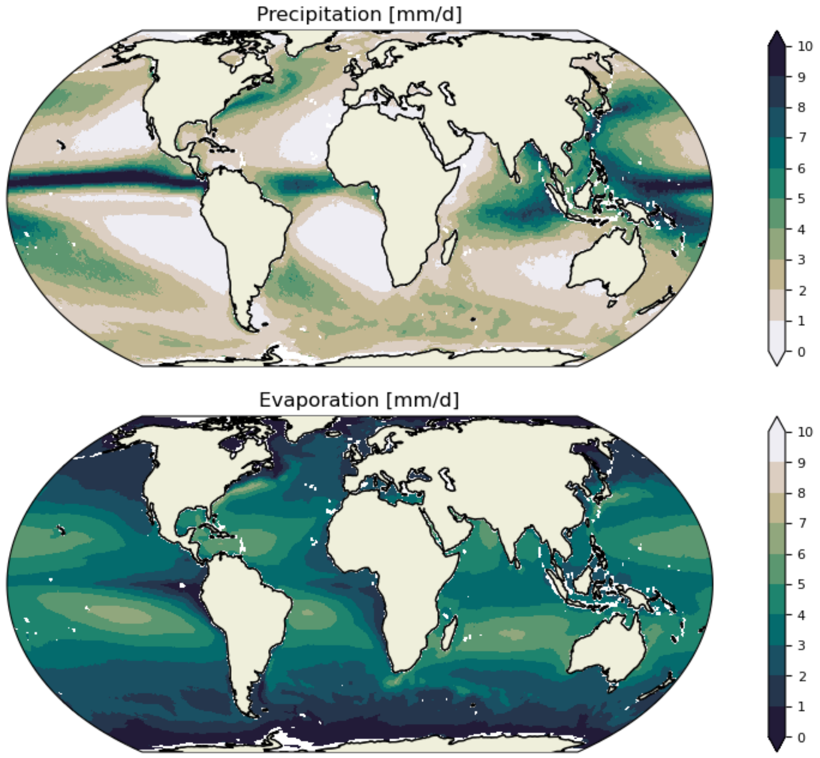


Figure 3.4: Precipitation and Evaporation fluxes computed from the NCEP/NCAR reanalysis for the period 2010-2019 *Kalnay et al.* (1996).

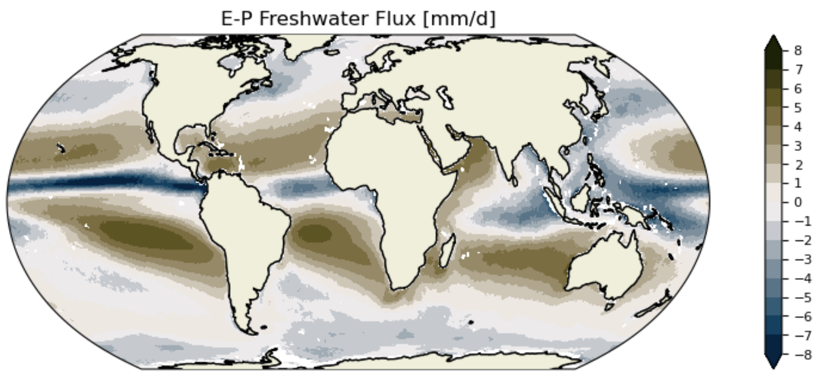


Figure 3.5: Net freshwater fluxes computed from the NCEP/NCAR reanalysis for the period 2010-2019 *Kalnay et al.* (1996).

3.3 Air-sea forcing of surface density: Surface buoyancy flux

The combination of surface heat and freshwater fluxes alters the surface density: warming or freshening will lighten surface waters, while cooling and evaporation will increase surface density. The forcing of surface density is thus

$$\mathcal{D} = -\frac{\alpha}{C_p}\mathcal{Q} + \beta\rho_0 S(E-P) \tag{3.25}$$

with units of mass per unit area and unit time. $\alpha = -\rho^{-1}\partial_\theta\rho$ is the density expansion coefficient for temperature (in K^{-1}), $\beta = \rho^{-1}\partial_S\rho$ is the density contraction coefficient for salinity (in g Kg^{-1}). From the linearized equation of state

$$\rho \approx \rho_0[1-\alpha(T-T_0)+\beta(S-S_0)], \tag{3.26}$$

or

$$\frac{\Delta\rho}{\rho_0} \approx -\alpha\Delta T + \beta\Delta_S, \tag{3.27}$$

with $\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}$ and $\beta = \frac{1}{\rho}\frac{\partial\rho}{\partial S}$.

This surface forcing is also a buoyancy flux, which is practically more useful

$$\boxed{\mathcal{B} = -g\mathcal{D}/\rho_0 = \frac{g}{\rho_0}\left[\frac{\alpha}{C_p}\mathcal{Q} - \beta\,\rho_0 S(E-P)\right]}, \tag{3.28}$$

(remember that $b = -g\delta\rho/\rho_0$ is the buoyancy).

Given that

$$\begin{aligned}
C_p &= [J\,Kg^{-1}\,K^{-1}] \\
g &= [m\,s^{-2}] \\
\rho_0 &= [kg\,m^{-3}] \\
\text{Net surface Heat flux}(\mathcal{Q}) &= W\,m^{-2} = [J\,m^{-2}\,s^{-1}] \\
\text{Net surface Freshwater flux}(\mathcal{W}\,S) &= (P-E)S = [g\,Kg^{-1}\,m\,s^{-1}]
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{\alpha}{C_p}Q &= [K^{-1}]/[J\,Kg^{-1}\,K^{-1}][J\,m^{-2}\,s^{-1}] = Kg\,m^{-2}\,s^{-1} \\
\beta\rho_0\mathcal{W} &= [g\,Kg^{-1}][kg\,m^{-3}][g\,Kg^{-1}\,m\,s^{-1}] = Kg\,m^{-2}\,s^{-1}
\end{aligned}$$

Therefore the surface buoyancy flux has units:

$$\mathcal{B} = [m\,s^{-2}][Kg\,m^{-2}\,s^{-1}]/[kg\,m^{-3}] = [m^2 s^{-3}] \tag{3.29}$$

Which perturbation is more important in generating density changes? Thermal or saline? It depends on the ambient temperature. From the linearised equation of state, the relative contribution of ΔT and ΔS are measured by α and β, or their ratio α/β, for given temperature and salinity changes, which depends strongly with temperature (the ratio increases with temperature). Hence, a temperature perturbation will have a larger effect on density in warm waters, and a salinity perturbation becomes more important in controlling density changes in cold waters.

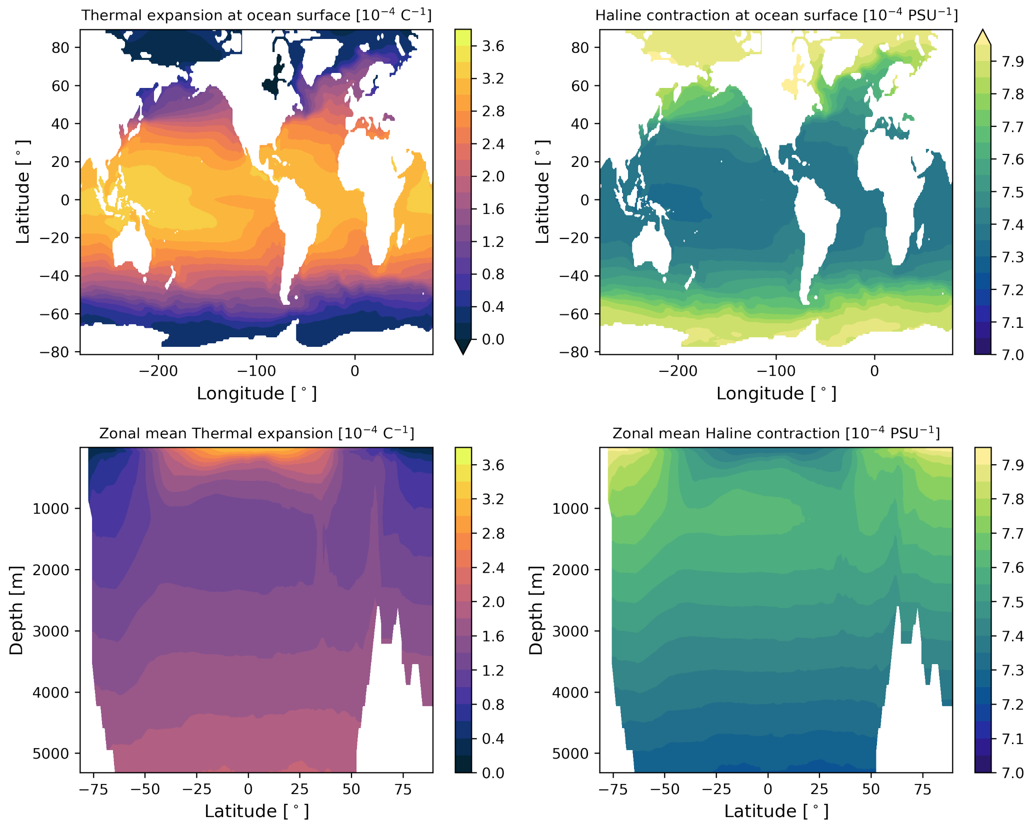


Figure 3.6: Time mean of the thermal expansion coefficient $\alpha = -\rho^{-1}\partial_\theta \rho$ and haline contraction coefficient $\beta = \rho^{-1}\partial_S \rho$.

3.4 Air-sea exchange of momentum

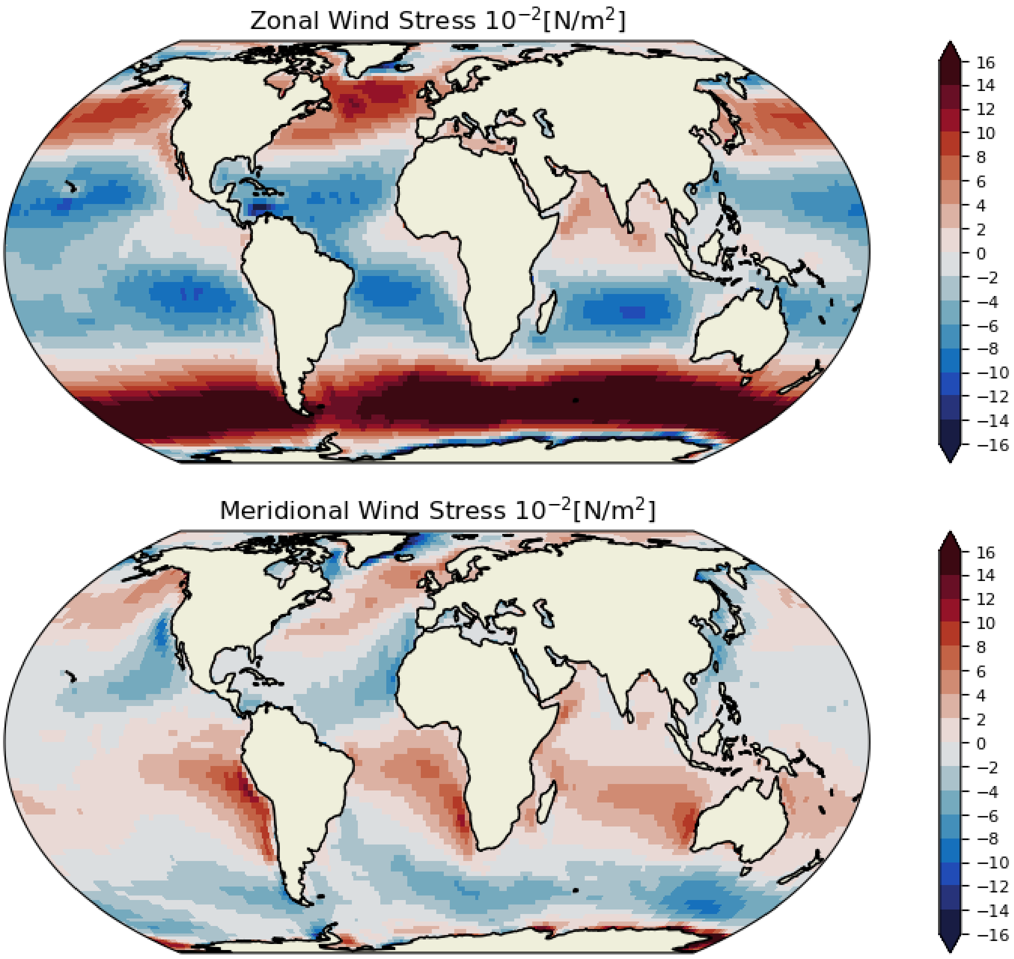


Figure 3.7: Surface zonal and meridional components of the wind stress computed from the NCEP/NCAR reanalysis for the period 2010-2019 *Kalnay et al.* (1996).

Chapter 4

Fundamental tools

This chapter will slowly grow with material from the GFD course.

4.1 Some basics

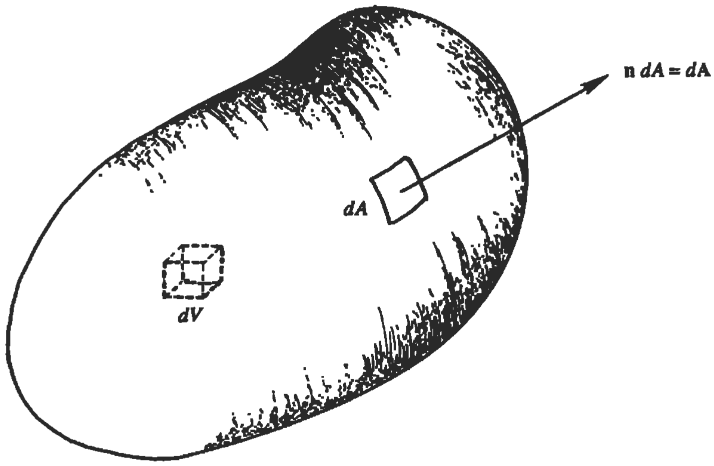
4.1.1 Shear stresses and Newton's law of viscosity

4.1.2 Pressure in a fluid

4.1.3 Tensors

4.1.4 (Gauss') Divergence theorem

The theorem relates a volume integral to a surface integral. Consider a volume V bounded by a closed surface A. Consider an infinitesimal surface element dA whose outward unit normal is $\mathbf{n}$. The vector ndA has magnitude dA and direction $\mathbf{n}$.



Gauss' theorem states that the volume integral of the divergence of **Q** is equal to the surface integral of the outflow of **Q**.

$$\int_v \frac{\partial Q}{\partial x_i} \mathrm{d}V = \int_A \mathrm{d}A_i Q \tag{4.1}$$

For a vector $\boldsymbol{Q}$:

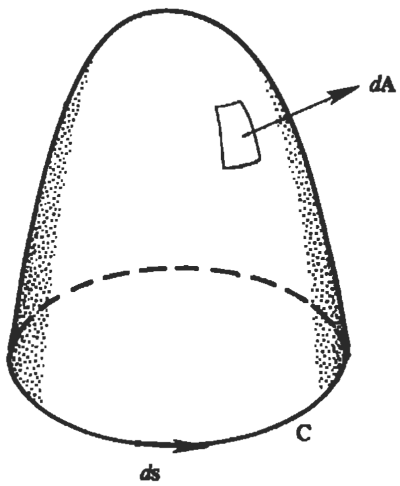
$$\boxed{\int_v \frac{\partial Q_i}{\partial x_i} \mathrm{d}V = \int_A \mathrm{d}A_i Q}, \tag{4.2}$$

which is now called the Divergence Theorem. In vector notation

$$\int_v \nabla \cdot \mathbf{Q} \, \mathrm{d}V = \int_A \mathrm{d}\mathbf{A} \cdot \mathbf{Q} \tag{4.3}$$

4.1.5 Stokes' theorem

The theorem relates a surface over an open surface to a line integral. Consider an open surface A, with bounding curve C. Let $d\mathbf{r}$ be an element of the bounding curve whose direction is that of the tangent.



Stokes' theorem states that

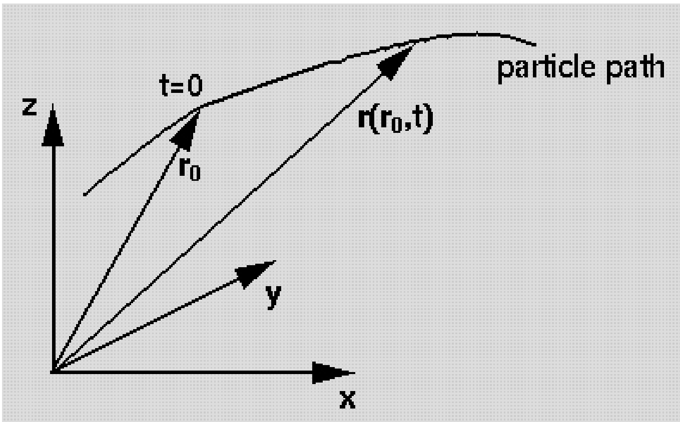
$$\boxed{\int_A (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}} \tag{4.4}$$

the surface integral of the curl of a vector field **F** is equal to the line integral of **F** along the bounding curve. The line integral of a vector around a closed curve C is the *circulation of the field about C*.

Prove that $div(curl\ \boldsymbol{u}) = 0$, for any vector $\boldsymbol{u}$.

4.2 Kinematics

- In the LAGRANGIAN description of motion, one essentially follows the history of an individual particle. A flow variable $F(r_0, t)$ and its velocity is given by $u_i = \mathrm{d}(r_i)/\mathrm{d}t$
- In the EULERIAN description one focuses on what happens at a spatial point r, so the flow variable is $F(r, t)$.
- In the Eulerian case, $\mathrm{d}/\mathrm{d}t$ gives the local rate of change of F at each point r and is not the total rate of change seen by a fluid particle ...



4.2.1 The material Derivative

We seek to calculate the rate of change of F at each point following a particle of fixed identity.

$$\boxed{\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}} \tag{4.5}$$

The material Derivative $\frac{DF}{Dt}$ is made of (1) the local rate of change at a given point (zero for steady flows...) and (2) the advective derivative.

$\frac{\partial F}{\partial t}$ is the local rate of change of F at a given point.

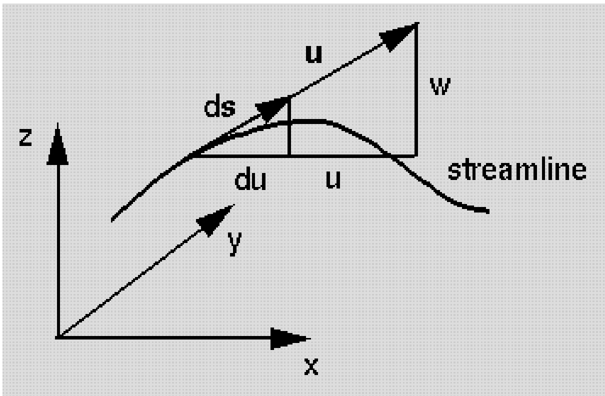
$u_i \frac{\partial F}{\partial x_i}$ is the advective derivative, it is the change in F as a result of advection of the particle from one location to another where F is different.

4.2.2 Streamlines and streamfunctions

The streamline

- At $t = t_0$, streamlines are curves that are tangent to direction of flow.
- For unsteady flows, streamlines change with time.

Let $ds = (dx, dy, dz)$ be an element of arc length along a streamline, and let $u = (u, v, w)$ be the local velocity vector along that streamline, then $dx/u = dy/v = dz/w$.



- Close to a solid boundary, streamlines are parallel to that boundary.
- The direction of the streamline is the direction of the fluid velocity.
- Fluid can not cross a streamline.
- Streamlines can not cross each other.
- Any particle starting on one streamline will stay on that same streamline.
- In unsteady flow, streamlines can change position with time.
- **Streamlines** are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction a fluid element will travel in at any point in time.
- **Pathlines** are the trajectories that individual fluid particles follow. These can be thought of as a "recording" of the path a fluid element in the flow takes over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.
- For a steady flow, the two are the same.

4.3 Equations of motion

There can be two kind of forces acting on fluids. Body forces, and we will restrict our attention, for now, to gravitational force per unit mass

$$\mathbf{g} = -\nabla(gz) = -\hat{k}\frac{\partial(gz)}{\partial z} = -\hat{k}g \tag{4.6}$$

and surface forces, which can be normal or tangential to the fluid. Normal forces will be relate to pressure, whereas tangential forces will be related to shear stresses.

In order to derive a principle of conservation of momentum we will start by applying Newton's law of motion to an infinitesimal element of fluid. The continuity equation, for an element of fluid of constant density is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{4.7}$$

and we multiply this by $\mathbf{u}$:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = f_x \tag{4.8}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = f_y \tag{4.9}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = f_z \tag{4.10}$$

(4.11)

which for a constant density reduces to

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = \mathbf{f} \tag{4.12}$$

If we express our body force per unit volume $\rho\mathbf{g}$, we arrive to the Cauchy equation of motion

$$\rho\frac{\mathrm{D}\,u_i}{\mathrm{D}\,t} = \rho g_i + \frac{\partial\tau_{ij}}{\partial x_j} \tag{4.13}$$

where the stress tensor τ_{ij} includes all surface forces. Using the constitutive equation for a Newtonian fluid, we now arrive to the Navier-Stokes equation

$$\rho\frac{\mathrm{D}\,u_i}{\mathrm{D}\,t} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u} \tag{4.14}$$

which reduces to the Euler equation under the assumption of frictionless flow

$$\rho \frac{\mathrm{D}\, u_i}{\mathrm{D}\, t} = \rho \mathbf{g} - \nabla p \tag{4.15}$$

4.3.1 Motion in a rotating frame of reference

Eq.4.30 is valid for an inertial or fixed frame of reference. But in GFD we measure positions and velocities relative to a frame of reference fixed on the surface of the Earth, which rotates w.r.t. to a frame inertial.

Let's have a frame of reference (x_1, x_2, x_3) rotating at a uniform angular velocity $\mathbf{\Omega}$ w.r.t. a fixed frame (X_1, X_2, X_3). Any vector $\mathbf{P}$ is represented by

$$\mathbf{P} = P_1 i_1 + P_2 i_2 + P_3 i_3 \tag{4.16}$$

For a fixed observer, the directions of the rotating unit vectors (i_1, i_2, i_3) change with time. The time derivatives of $\mathbf{P}$ is thus

$$\begin{aligned} \left(\frac{d\mathbf{P}}{dt}\right)_I = & \frac{d}{dt}(P_1 i_1 + P_2 i_2 + P_3 i_3) = \\ & i_1 \frac{dP_1}{dt} + i_2 \frac{dP_2}{dt} + i_3 \frac{dP_3}{dt} + P_1 \frac{di_1}{dt} + P_2 \frac{di_2}{dt} + P_3 \frac{di_3}{dt} \end{aligned} \tag{4.17}$$

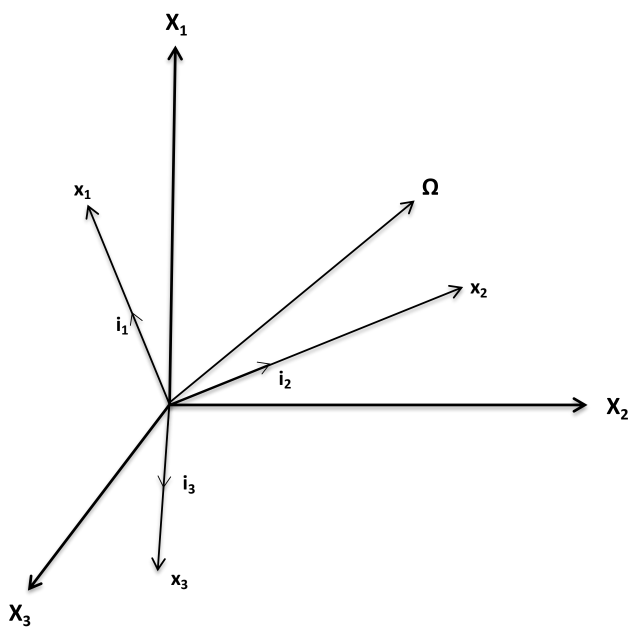


Figure 4.1: *Coordinate frame (x_1, x_2, x_3) rotating at angular velocity Ω with respect to a fixed frame (X_1, X_2, X_3).*

For an observer rotating with (x_1, x_2, x_3) the rate of change of $\mathbf{P}$ is equal to the first three terms in (4.17), and so

$$\left(\frac{d\mathbf{P}}{dt}\right)_I = \left(\frac{d\mathbf{P}}{dt}\right)_R + P_1\frac{di_1}{dt} + P_2\frac{di_2}{dt} + P_3\frac{di_3}{dt} \tag{4.18}$$

Each unit vector i traces a cone with radius $\sin\alpha$, where α is a constant angle. In a small interval of time δt, i rotates through a small angle $\delta\theta$

$$\delta i = \sin\alpha\delta\theta, \tag{4.19}$$

which is the length travelled by the top of i as perceived in the inertial frame.

The rate of change of the angle $\delta\theta$ is just the angular velocity so that $\delta\theta = |\mathbf{\Omega}|\delta t$ and

$$\delta i = |\mathbf{\Omega}| \sin\alpha \, \delta t. \tag{4.20}$$

Using the definition of the vector cross product, and noting that direction of the rate of change is perpendicular to the plane $(\mathbf{\Omega}, \mathbf{i})$

$$\left(\frac{d\mathbf{i}}{dt}\right)_I = \mathbf{\Omega} \times \mathbf{i} \tag{4.21}$$

is the rate of change for any rotating vector $\mathbf{i}$ as perceived in the inertial frame.

From (4.18), in a small time δt the change in $\mathbf{P}$ as seen in the rotating frame is given by the relation

$$(\delta\mathbf{P})_I = (\delta\mathbf{P})_R + (\delta\mathbf{P})_{rot}, \tag{4.22}$$

and the rate of change of the vector $\mathbf{P}$ in the inertial and rotating frames are related by

$$\boxed{\left(\frac{d\mathbf{P}}{dt}\right)_I = \left(\frac{d\mathbf{P}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{P}} \tag{4.23}$$

Applying this rule to the position vector $\mathbf{r}$

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{4.24}$$

or

$$\mathbf{u}_I = \mathbf{u}_R + \mathbf{\Omega} \times \mathbf{r} \tag{4.25}$$

Applying this rule to the velocities

$$\left(\frac{d\mathbf{u_I}}{dt}\right)_I = \left(\frac{d\mathbf{u_I}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{u_I} \tag{4.26}$$

$$\begin{aligned}\left(\frac{d\mathbf{u_I}}{dt}\right)_I &= \frac{d}{dt}\left(\mathbf{u}_R + \mathbf{\Omega} \times \mathbf{r}\right)_R + \mathbf{\Omega} \times \left(\mathbf{u}_R + \mathbf{\Omega} \times \mathbf{r}\right) \\ &= \left(\frac{d\mathbf{u_R}}{dt}\right)_R + \mathbf{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{u}_R + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\end{aligned} \tag{4.27}$$

Hence, accelerations in the two frames are related as

$$\left(\frac{d\mathbf{u_R}}{dt}\right)_R = \left(\frac{d\mathbf{u_I}}{dt}\right)_I - 2\mathbf{\Omega} \times \mathbf{u}_R - \mathbf{\Omega}^2 \mathbf{r}_\perp \tag{4.28}$$

The second term of the r.h.s is the Coriolis force and the last term the centrifugal force per unit mass. This last term is added to the Newtonian gravity as an effective gravity

$$\mathbf{g} = \mathbf{g_n} + \mathbf{\Omega}^2 \mathbf{r}_\perp \tag{4.29}$$

The apparent force $\mathbf{\Omega}^2 \mathbf{r}_\perp$ will be zero at the poles.

The momentum equations are now

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} - (2\mathbf{\Omega} \times \mathbf{u}) \tag{4.30}$$

It is clear that the Coriolis force $(-2\mathbf{\Omega} \times \mathbf{u})$ will deflect a particle to the right of its direction in the northern hemisphere (right-hand rule). As the Coriolis force constantly acts normal to the fluid path, it will not accelerate the particle (in fact, Coriolis does not play any role in the energy equation).

4.3.2 Thin shell approximation

A scale analysis of the continuity equation reveals that, for typical length scales much larger than typical vertical scales, $L \gg H$, horizontal velocities must be much larger than the vertical ones, $U \gg W$.

Now, decomposing the angular velocity vector into its three components (Fig.4.2), we have

$$\begin{aligned}\Omega_x &= 0 \\ \Omega_y &= \Omega \cos\theta \\ \Omega_z &= \Omega \sin\theta\end{aligned}$$

The Coriolis term, assuming $U \gg W$, has the following components

$$2\mathbf{\Omega} \times \mathbf{u} = 2\Omega \left[(-v \sin\theta)\hat{\imath} + (u \sin\theta)\hat{\jmath} - (u \cos\theta)\hat{k} \right] \tag{4.31}$$

and defining the Coriolis parameter as $f = 2\Omega \sin\theta$, which is now clearly twice the angular velocity and hence a (planetary) vorticity

$$2\mathbf{\Omega} \times \mathbf{u} = (-fv)\hat{\imath} + (fu)\hat{\jmath} - (2\Omega u \cos\theta)\hat{k} \tag{4.32}$$

But the vertical component of the Coriolis force, $2\Omega u \cos\theta$, is negligible compared to the dominant terms in the vertical equation of motion, namely the pressure gradient and the gravitational acceleration. Our final set of momentum equations reduces to

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u \tag{4.33}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v \tag{4.34}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu \nabla^2 w \tag{4.35}$$

Or

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}} \tag{4.36}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

4.3.3 The β-plane

A first approximation is to set the Coriolis parameter, f, to a constant value. This approximation, denoted the f-plane, is useful in some very idealized studies when the westward propagation of disturbances is not of interest or it is purposely neglected. But for large-scale dynamics it is not appropriate, when flows occurring over large horizontal scales are of interest. Rossby waves depend on variations of f, it is their restoring mechanism, for example, and the large-scale dynamics of the ocean will this be affected by latitudinal variations in the Coriolis parameter. An approximation can be done, however, to make equations more tractable, and it consists of considering a cartesian plane over which f does vary, so neglecting spherical coordinates.

The β-plane approximation is useful to avoid the sphericity and staying in a cartesian plane, yet retaining the dynamical effects of sphericity itself.

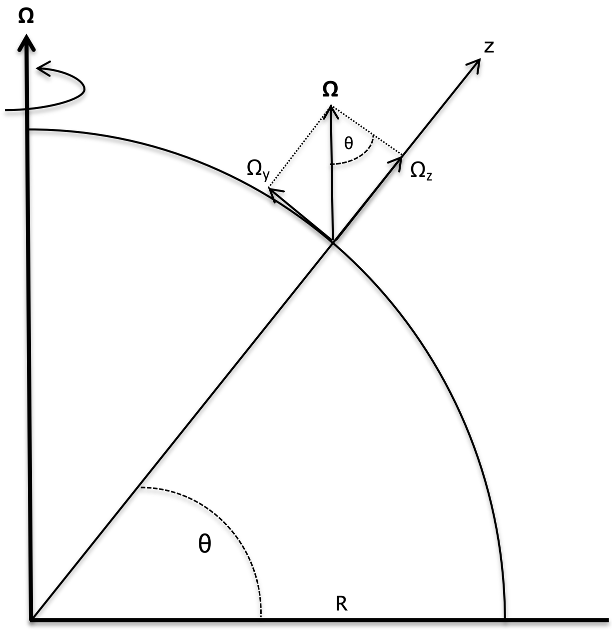


Figure 4.2: *Components of the angular velocity vector for a point on the sphere.*

The plane $z = 0$, what will be called the β-plane is tangent to the sphere in (a, θ_0, ϕ_0). For small variations in latitude we can approximate $tan(\theta - \theta_0) \approx \theta - \theta_0$. hence, our meridional cartesian coordinate is

$$\begin{aligned} y &= a(\theta - \theta_0) \\ z &= r - a \\ x &= (\phi - \phi_0) a\, cos\theta_0, \end{aligned}$$

where r is the distance of the fluid from the center of the sphere, θ is the latitude, ϕ the longitude, and a is the radius of the Earth.

Hence, latitude θ is a linear function of y

$$\theta = \theta_0 + \frac{y}{a}. \tag{4.37}$$

Now, for small variations in latitude we have:

$$sin\theta \approx sin\theta_0 + cos\theta_0 \frac{y}{a}, \tag{4.38}$$

as a truncated series around θ_0. And we can express f as the following:

$$f = 2\Omega sin\theta = 2\Omega sin\theta_0 + \frac{2\Omega}{a} cos\theta_0\, y = f_0 + \beta\, y. \tag{4.39}$$

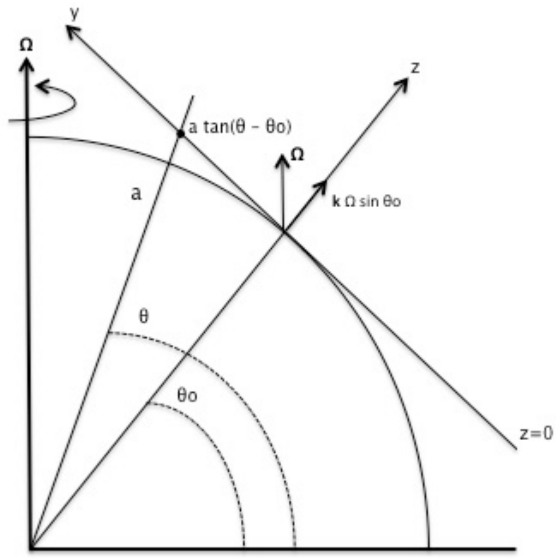


Figure 4.3: *A cartesian reference system (x,y,z) and its associated spherical system (r, θ, ϕ) around the point (a, θ_0, ϕ_0). The plane $z = 0$ (or β-plane) is tangent to the sphere around the point (a, θ_0, ϕ_0). The approximation $tan(\theta - \theta_0) \approx (\theta - \theta_0)$ is well justified for small variations in latitude. On the β-plane, the rotation vector is $\mathbf{k}\Omega\, sin\theta$, where $sin\theta \approx sin\theta_0 + (y/a)cos\theta_0$.*

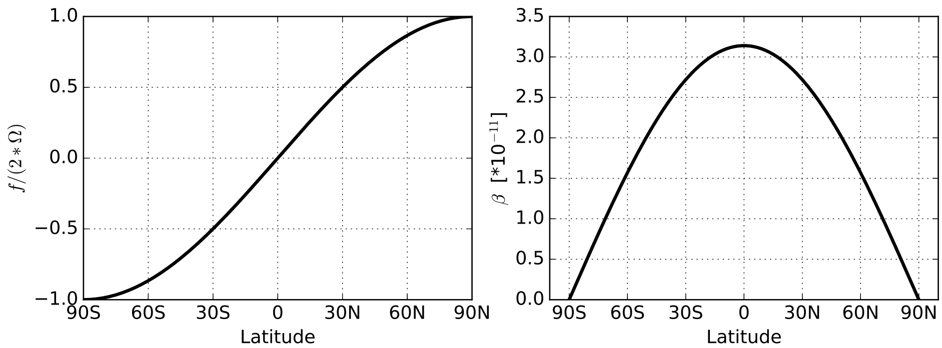


Figure 4.4: *The Coriolis parameter f and its meridional gradient β as a function of latitude.*

Where we have introduced $\beta = \frac{2\Omega}{a} cos\theta_0$.

But what is β? We have gone from $f = 2\Omega sin\theta$ to $f = f_0 + \beta y$. The dependence of f on latitude is conserved because of the linear relationship between f and y. This is an important result: **we are not working on spherical coordinates but the dynamical effects of sphericity are retained.**

β is called the gradient of planetary vorticity given that:

$$\frac{\partial f}{\partial y}(\theta = \theta_0) = \frac{1}{a}\frac{\partial f}{\partial \theta}(\theta = \theta_0) = \frac{2\Omega}{a} cos\theta_0 = \beta. \tag{4.40}$$

Typical mid-latitude values for f and β are 10^{-4} s^{-1} and 10^{-11} m^{-1}s^{-1} (Fig. 4.4).

In conclusion, we have the β-plane approximation as

$$\boxed{f = f_0 + \beta y} \tag{4.41}$$

For relatively large areas, with θ varying over a few tens of degrees, between mid-latitudes and the equator, the tangent plane approximation is called β-plane. This approximation is only valid if

$$\beta y \ll f_0 \quad or \quad \frac{\beta y}{f_0} \ll 1. \tag{4.42}$$

For even smaller variations in θ the f-plane is used, where

$$f = f_0 = 2\Omega\, sin\theta_0. \tag{4.43}$$

4.4 Kinematical and dynamical approximations

4.4.1 Hydrostatic balance

The vertical component (the component parallel to the gravitational force, $\mathbf{g}$) of the momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{4.44}$$

where w is the vertical component of the velocity and $\mathbf{g} = -g\mathbf{k}$. If the fluid is static the gravitational term is balanced by the pressure term and we have

$$\frac{\partial p}{\partial z} = -\rho g, \tag{4.45}$$

which is called the *hydrostatic balance*, or hydrostasy. Scaling analysis shows that the hydrostatic balance is the dominant balance within the vertical momentum equation , so long as the vertical length scales of motion are much smaller than the horizontal length scales. Such scales are relevant for large-scale ocean climate modeling, and global ocean models typically assume a hydrostatic balance, and this constitutes a basic assumption of the primitive equations. Integrating the hydrostatic balance vertically from the ocean surface η determines the pressure at a point in the ocean column

$$p(z) = p_a + g\int_z^{\eta} \mathrm{d}z'\rho(z'), \tag{4.46}$$

where p_a is the sea surface pressure resulting from external forcing (e.g., atmospheric loading, sea ice, ...).

Scaling and aspect ratio

For a Boussinesq fluid, the momentum equations are

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f}\times\mathbf{u} = -\nabla\phi \tag{4.47}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\phi}{\partial z} + b, \tag{4.48}$$

where $\phi = p/\rho_0$ and buoyancy $b = -g\rho/\rho_0$. In the case of $f = 0$ the horizontal momentum equation reduces to

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla\phi \tag{4.49}$$

and a scaling for the horizontal equation is

$$\frac{U}{T} \sim \frac{\Phi}{L}, \text{or } \frac{LU}{T} \sim \Phi, \text{or } U^2 \sim \Phi. \tag{4.50}$$

Using mass conservation to scale vertical velocities we obtain

$$\nabla_z \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0. \tag{4.51}$$

A scaling of this equation is

$$\frac{U}{L} + \frac{W}{H} = 0 \tag{4.52}$$

$$W = \frac{H}{L} U = \alpha U \tag{4.53}$$

where $\alpha \equiv \frac{H}{L}$ is the aspect ratio between the typical horizontal and vertical scales. The advective terms in the vertical momentum equation scale as

$$\frac{\mathrm{D} w}{\mathrm{D} t} \sim \frac{W}{T} = \frac{U}{L} W = \frac{U}{L}(\frac{H}{L} U) = \frac{U^2 H}{L^2}. \tag{4.54}$$

Now we can use the scaling for the horizontal and vertical motions, together with the aspect ratio of their typical scales, to reveal the condition for hydrostasy.

For hydrostatic balance to hold, the ratio of advective terms to the pressure gradient term in (4.48) must be

$$\frac{|\frac{\mathrm{D} w}{\mathrm{D} t}|}{|\frac{\partial \phi}{\partial z}|} \ll 1 \tag{4.55}$$

This implies that

$$\frac{|\frac{\mathrm{D} w}{\mathrm{D} t}|}{|\frac{\partial \phi}{\partial z}|} \sim \frac{U^2 H / L^2}{U^2 / H} \sim \left(\frac{H}{L}\right)^2 \ll 1. \tag{4.56}$$

In other words, the aspect ratio should be

$$\alpha^2 \equiv \left(\frac{H}{L}\right)^2 \ll 1 \tag{4.57}$$

for the advective terms in the vertical momentum to be neglected. The hydrostatic balance is then a *small aspect ratio approximation*.

4.5 Static instability, the parcel method and Buoyancy frequency

Consider a stratified ocean and a parcel of fluid initially at rest, and therefore in hydrostatic balance. We will focus on vertical displacements and the restoring force is gravity. Consider a small adiabatic displacement of the parcel upward by δz, without altering the background pressure field. If the parcel is now lighter then the local environment, it will feel an upward pressure gradient force larger than the downward gravitational force, it will accelerate upwards and will become buoyant. In this case the fluid is statically unstable. If, instead, the parcel finds itself heavier than its sorroundings, the downward gravitational force will be greater than the upward pressure force, the fluid will sink back to its original position and will oscillate. This condition is statically stable.

Consider an incompressible fluid in which the density of the displaced parcel is conserved, $D\rho/Dt = 0$. If the environmental profile is $\tilde{\rho}(z)$ and the density of the parcel is ρ, a parcel displaced to a level $z + \delta z$ will show a change in density with respect to the local environment equal to

$$\delta\rho = \rho(z+\delta z) - \tilde{\rho}(z+\delta z) = \tilde{\rho}(z) - \tilde{\rho}(z+\delta z) = -\frac{\partial\tilde{\rho}}{\partial z}\delta z, \tag{4.58}$$

where the derivative on the right-hand side is the environmental gradient of density.

If $\frac{\partial\tilde{\rho}}{\partial z} < 0$, the parcel will be heavier than its sorroundings and will sink back in a stable condition.

If $\frac{\partial\tilde{\rho}}{\partial z} > 0$, the parcel will be buoyant in a statically unstable fluid.

That is, the stability of a parcel of fluid is determined by the gradient of the environmental density.

The upward force, per unit volume, on the displaced parcel is

$$F = -g\delta\rho = g\frac{\partial\tilde{\rho}}{\partial z}\delta z \tag{4.59}$$

and the equation of motion of the fluid parcel is thus

$$\rho(z)\frac{\partial^2\delta z}{\partial t^2} = g\frac{\partial\tilde{\rho}}{\partial z}\delta z, \tag{4.60}$$

or

$$\frac{\partial^2\delta z}{\partial t^2} = \frac{g}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial z}\delta z. \tag{4.61}$$

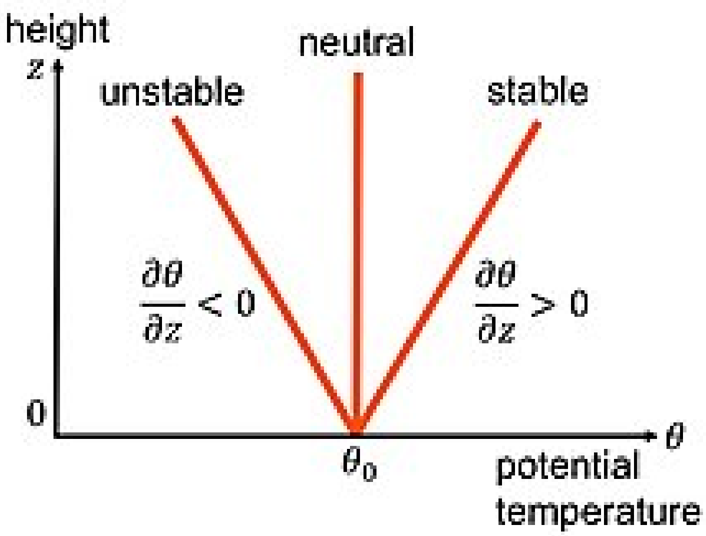


Figure 4.5: Possible temperature vertical profiles, in the atmosphere or ocean, giving rise to unstable, neutral or stable conditions.

Static stability measures how quickly a water parcel is restored to its position in the water column if displaced vertically. If unstable, the water column has the potential to overturn.

In stable water column conditions ($\frac{\partial\tilde{\rho}}{\partial z} < 0$), the parcel experiences a restoring force and will oscillate at a given frequency:

$$\frac{\partial^2 \delta z}{\partial t^2} = -N^2 \delta z, \qquad (4.62)$$

where

$$N^2 = -\frac{g}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial z}, \qquad (4.63)$$

and N is the Brunt-Vaisala frequency. In liquids, it is a good approximation to replace $\tilde{\rho}$ by ρ_0.

If $N^2 < 0$, the density profile is unstable, the parcel continues to ascend and convection occurs. This is the condition for convective instability. Convection causes fluid parcels to mix and reduces an unstable profile to neutral stability.

Question: What is happening with global warming to the stratification of the ocean? How can you modify the stratification of the ocean? Which could be the implications?

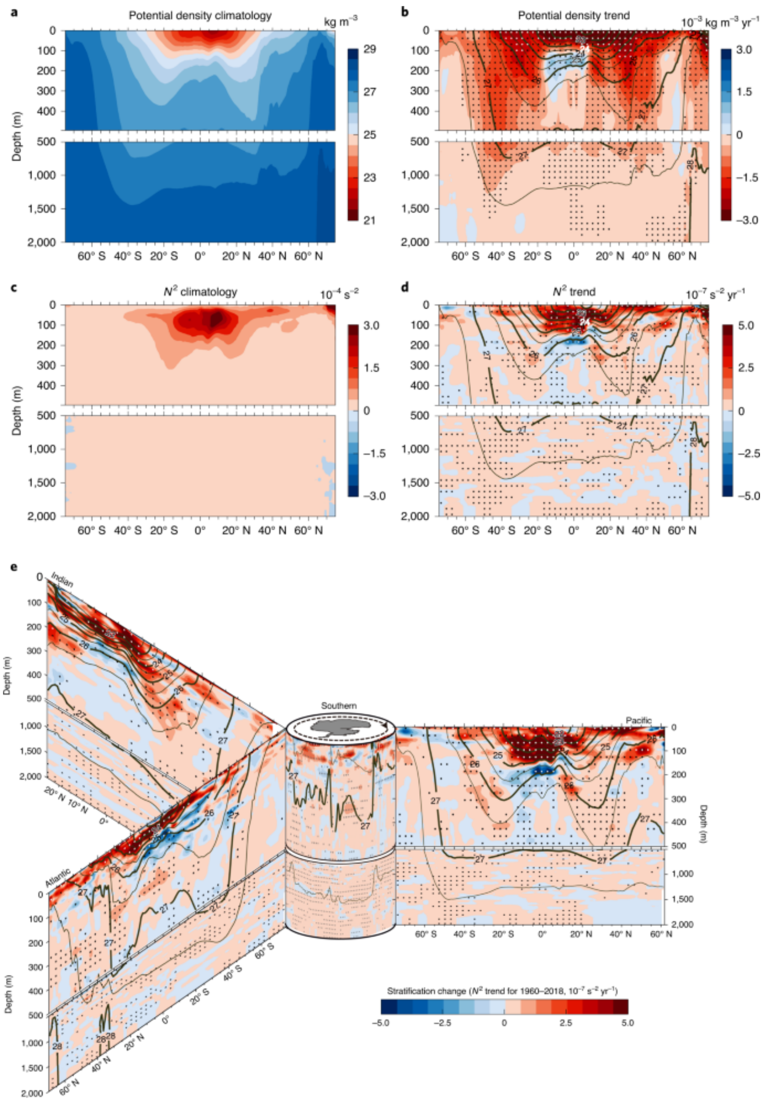


Figure 4.6: (a) Climatological potential density in the ocean, (b) its annual trend, (c) climatological stratification and (d) and its annual trend. Data are from a multiple-source observations reconstruction (*Li et al. Increasing ocean stratification over the past half-century. Nat. Clim. Chang. 10, 1116-1123 (2020)*).

Seawater generally forms stratified layers with lighter (warmer) waters near the surface and denser waters at greater depth. This stable configuration acts as a barrier to water mixing that impacts the efficiency of vertical exchanges of energy, carbon, oxygen and other constituents.

Stratification globally has increased by a substantial 5.3% between 1960-2018. Increasing stratification has important climate implications. For example, the expected decrease in ocean ventilation could affect ocean heat and carbon uptake, water mass formation and transformation. Also, there are implications for density-driven ocean circulation and, in particular, the Atlantic Meridional Overturning Circulation, which already shows some evidence of slowdown.

4.6 Rossby number

We consider the dynamical balance in the horizontal components of the momentum equation. In the horizontal plane (along geopotential surfaces) we find that the Coriolis term is much larger than the advective terms and the dominant balance is between Coriolis and the horizontal pressure force. The balance is called *geostrophic balance*, and it occurs when the Rossby number is small.

The horizontal momentum equation is

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla_z p, \tag{4.64}$$

where $\mathbf{v} = (u, v, w)$ and $\mathbf{u} = (u, v, 0)$. A scaling analysis of the second (U^2/L) and third (fU) terms, where U is the approximate magnitude of the horizontal velocities and L is a typical length scale over which that velocity varies, reveals the importance of rotation. The ratio of the sizes of the advective and Coriolis terms defines the Rossby number:

$$\boxed{Ro \equiv \frac{U}{fL}} \tag{4.65}$$

The Rossby number characterizes the importance of rotation in a fluid. It is the ratio of the magnitude of the relative acceleration to the Coriolis acceleration, and it is of fundamental importance in geophysical fluid dynamics.

4.7 Geostrophic and Thermal Wind Balance

If the Rossby number is sufficiently small, then the rotation term dominates the nonlinear advection term, and if the time period of the motion scales advectively (or there are no accelerations) then the rotation term also dominates the local time derivative. The only term that can then balance the rotation term is the pressure term, leaving us with

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4.66}$$

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{4.67}$$

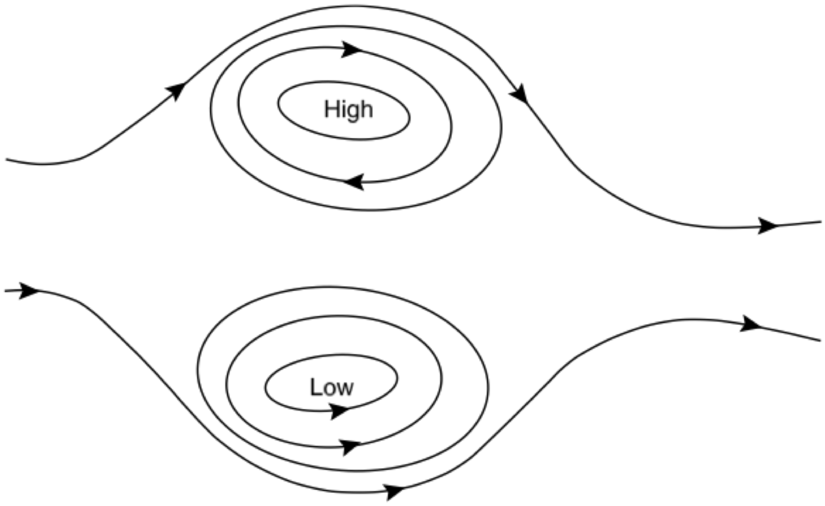


Figure 4.7: Schematic of a geostrophically balanced flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure. Cyclonic flow is anticlockwise around a low pressure region. [from *Vallis* (2006)]

This balance is known as *geostrophic balance*, and is one of the pillars of geophysical fluid dynamics. We can now define geostrophic velocities as

$$fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad fv_g = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4.68}$$

and for flows with a low Rossby numbers, $u \approx u_g$ and $v \approx v_g$.

A geostrophic flow is parallel to lines of constant pressure (isobars). If $f > 0$, after a pressure gradient is initiated somehow, the fluid starts to move down the gradient. Then, the fluid experiences the Coriolis force to the right and therefore swings to the right. The fluid eventually moves along isobars (along the slope, not down it), with the pressure force down the slope balanced by the Coriolis force up the slope. In the northern hemisphere, the flow is anticlockwise round a region of low pressure and clockwise around a region of high pressure.

Consider now a plane horizontal flow in which density does not vary along the fluid path (the Boussinesq approximation). In this case the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.69}$$

We can now define a function $\psi(x, y, t)$ such that

$$u \equiv -\frac{\partial \psi}{\partial y}, \quad (4.70)$$

$$v \equiv \frac{\partial \psi}{\partial x}, \quad (4.71)$$

and Eq.4.69 is thus satisfied, and this is called a *streamfunction*.

Returning to our geostrophic balance, if the Coriolis force is constant and if density does not vary in the horizontal, the geostrophic flow is horizontally non-divergent

$$\nabla_z \cdot \mathbf{u_g} = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0, \quad (4.72)$$

and we may define a geostrophic streamfunction, ψ_g, by $\psi_g \equiv \frac{p}{f\rho}$, and

$$u_g \equiv -\frac{\partial \psi}{\partial y}, \quad v_g \equiv \frac{\partial \psi}{\partial x}. \quad (4.73)$$

Thermal wind

Thermal wind balance arises when combining the geostrophic and hydrostatic approximations. They are useful in elucidating how temperature differences in the horizontal can lead to vertical variations in geostrophic velocities, hence the term *thermal wind equations*.

Taking the vertical derivative of the geostrophic equations for a Boussinesq fluid

$$\rho_0 f \partial_z u = -\partial_z \frac{\partial p}{\partial y} \quad (4.74)$$

$$\rho_0 f \partial_z v = \partial_z \frac{\partial p}{\partial x}. \quad (4.75)$$

Combining these with the hydrostatic balance, $\partial_z p = -\rho g$, and changing the order of differentiation for p, gives

$$\rho_0 f \, \partial_z u = g \, \partial_y \rho \quad (4.76)$$

$$\rho_0 f \, \partial_z v = -g \, \partial_x \rho. \quad (4.77)$$

These equations represent the *thermal wind balance*, and the vertical derivative of the geostrophic wind is the 'thermal wind'. Thermal wind balance

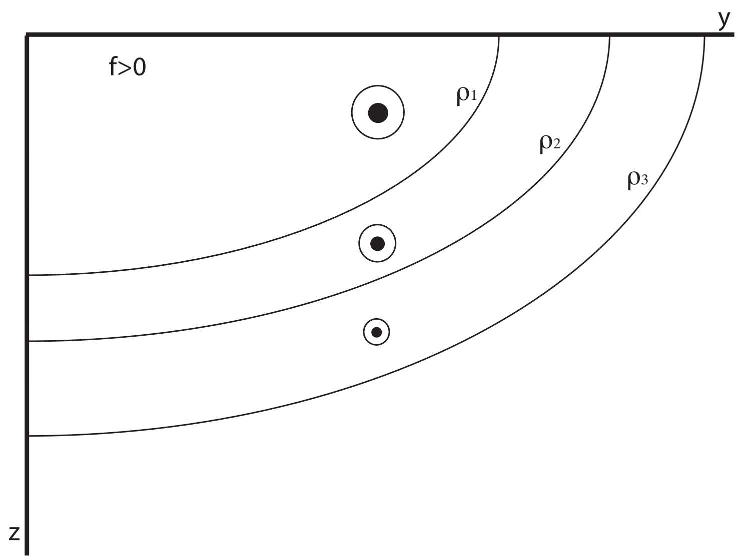


Figure 4.8: Schematic of thermal wind balance in the northern hemisphere. Shown are surfaces of constant density, or isopycnals. Density increases with depth and latitude, $\rho_3 > \rho_2 > \rho_1$. The termal wind associated with this density field is eastward, or out of the page, and decreases with depth. The same eastward thermal wind velocity would have resulted in the southern hemisphere, with $\rho_y < 0$ and $f < 0$.

says that **the geostrophic velocity has a vertical *thermal wind shear* in case where density has a horizontal gradient**.

In general, zonally averaged ocean temperature decrease poleward due to the differential heating received from solar radiation. Neglecting salinity effects on density, this poleward reduction in temperature corresponds to a poleward increase in density. Also, for a stably stratified fluid, density increases with depth. In a zonally-averaged flow, $\partial_x \rho = 0$, and so thermal wind reduces to

$$\partial_z u = \frac{g}{\rho_0 f} \partial_y \rho \tag{4.78}$$

This equation is telling us that, if temperature falls in the poleward direction, $\partial_y \rho > 0$, then the zonally-averaged thermal wind is eastward. Wind shear also increases as we move upward in the ocean, $\partial_z u > 0$, which yields a surface intensified zonal velocity field. Thermal wind, although diagnostic, represents a valid steady state balance of a frictionless rotating fluid. That is, in the presence of rotation, a flow can exist in steady state with nonflat isopycnals. Vertical integration of the thermal wind relation, along with knowledge of the geostrophic velocity at a point along the integration path, allows for determination of the full geostrophic velocity in terms of density. However, the baroclinic density field (with a horizontal gradient) is related to the baroclinic component of the velocity field through thermal wind balance. The barotropic flow component has zero vertical shear.

4.8 The Rossby radius

Consider a field that is in geostrophic balance. Let us assume that the pressure field is perturbed by some forcing of the horizontal scale L so that the flow is momentarily out of geostrophic balance. The subsequent process by which the system comes into some new balance is called "geostrophic adjustment". How this occurs depends on the relative scale of the disturbance L compared to an intrinsic scale of motion of a rotating ocean (or atmosphere). This intrinsic scale is the Rossby Radius of Deformation L_d. L_d is the length scale at which rotational effects become as important as buoyancy or gravity wave effects in the evolution of the flow about some disturbance.

The Rossby radius of deformation is a length scale of fundamental importance in atmosphere-ocean dynamics. For example, in the first stage of an adjustment problem, first the disturbance has a small structure and gravity dominates with a very large pressure gradient. Later, as the perturbation spreads over a larger horizontal scale, Coriolis becomes more important and of similar magnitude as the pressure gradient, and thus rotation causes a response that is much different from a non-rotating case.

Using a geostrophic flow, it is easy to show that the Rossby radius of deformation, L_d, is

$$L_d = c/|f| = (gH)^{1/2}/|f| \tag{4.79}$$

where c is the phase speed of the gravity wave. For the deep ocean, where H= 4 km and c= 200 m/s, the Rossby radius is about 2000 km. Which is much larger than depth, so the hydrostatic approximation is valid. However, the ocean is not only in rotation but also stratified, and so what is more important is not the barotropic radius of deformation but rather the baroclinic ones

$$L_d = c_n/|f| \tag{4.80}$$

where c_n are baroclinic gravity wave phase speeds. So the Rossy radius is directly related to the phase speed of long, baroclinic gravity waves, which is also a very useful parameter in the study of ocean wave dynamics. A global atlas of the first baroclinic gravity-wave phase speed, c, has been computed on a 1-degree global grid from observations (Fig. 4.9) as follows

$$c_n \sim \frac{1}{n\pi} \int_{-H}^{0} N \, \mathrm{d}z \tag{4.81}$$

where N is the buoyancy frequency.

Now, the first baroclinic Rossby radius, given that c_1=1-3 m/s, is L_d ~10-30 km with values increasing towards low latitudes (Fig. 4.10). Mesoscale eddies have the size of the first baroclinic Rossby radius, therefore in order to resolve mesoscale eddies and associated fluxes, ideally an ocean model should have at least two grid points within L_d. It is clear from Fig. 4.11 that standard global ocean model can resolve mesoscale fluxes up to ~25°, poleward of that latitude fluxes need to be parameterized. Benefits of having fine-resolution ocean models is illustrated in Fig. 4.12, where eddies and filaments are ubiquitous in the fine-resolution version of the model whereas a laminar ocean is simulated in the (standard!) 1° version.

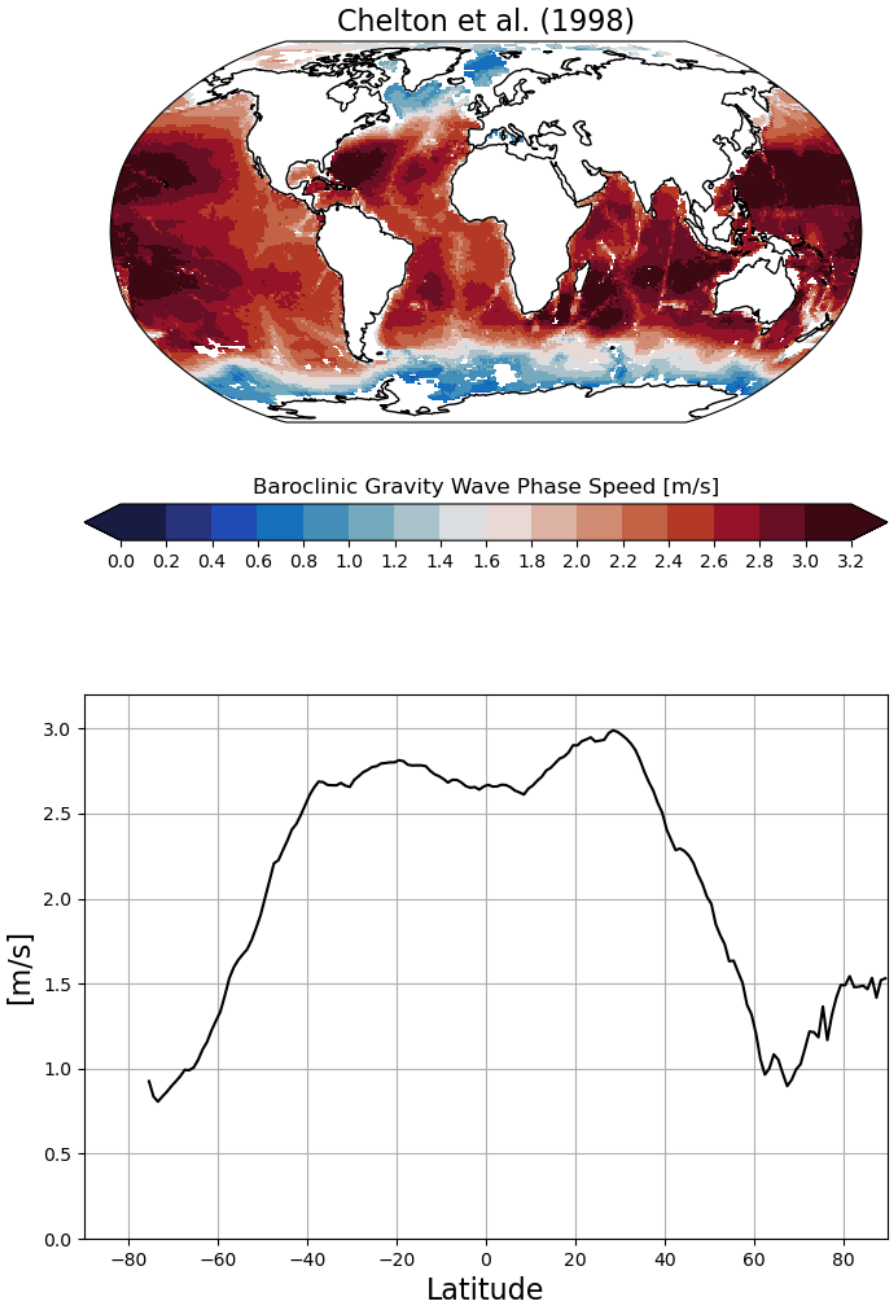


Figure 4.9: A global map of the first baroclinic gravity wave phase speed and its zonal mean. [data from Chelton et al., 1998]

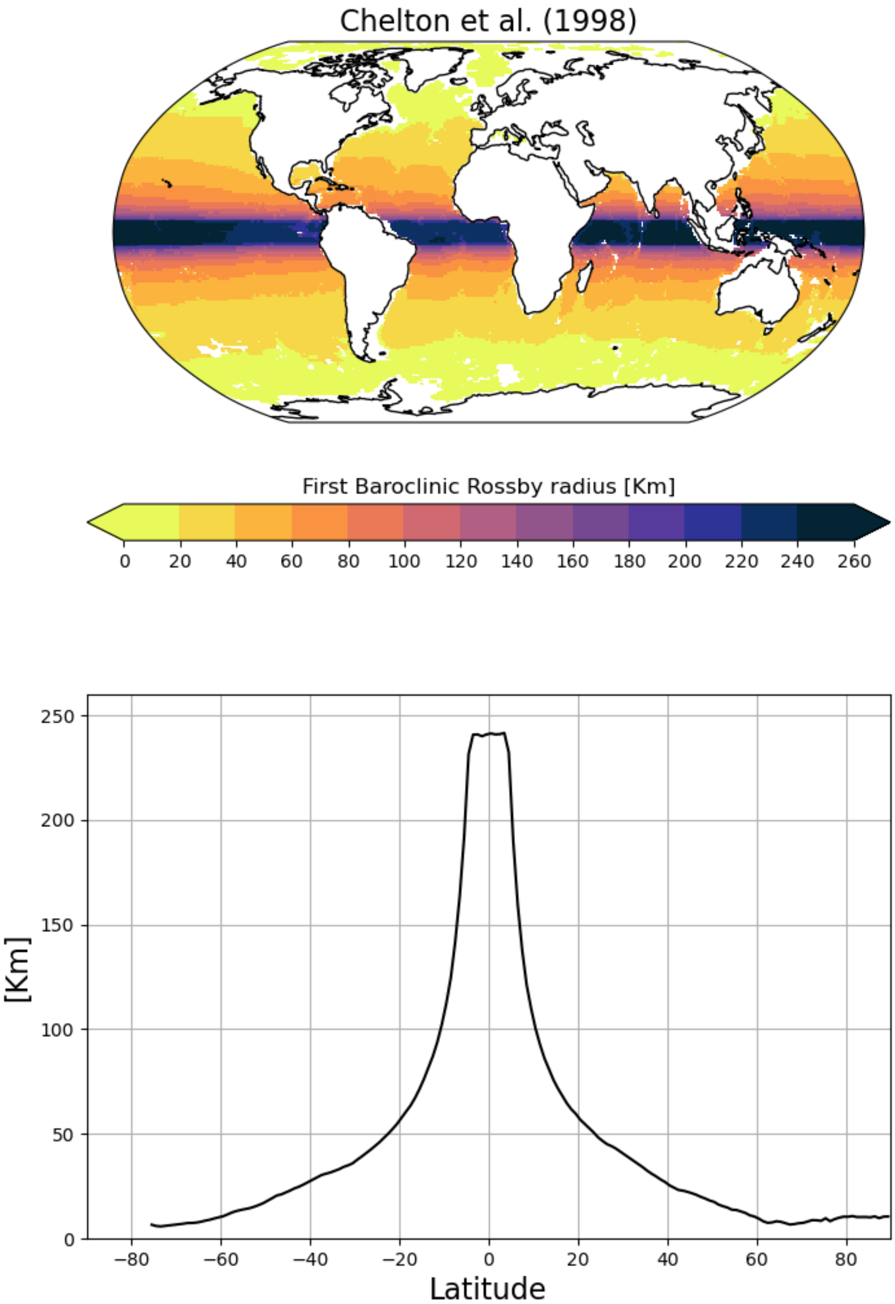


Figure 4.10: A global map of the first baroclinic Rossby radius of deformation and its zonal mean. [data from Chelton et al., 1998]

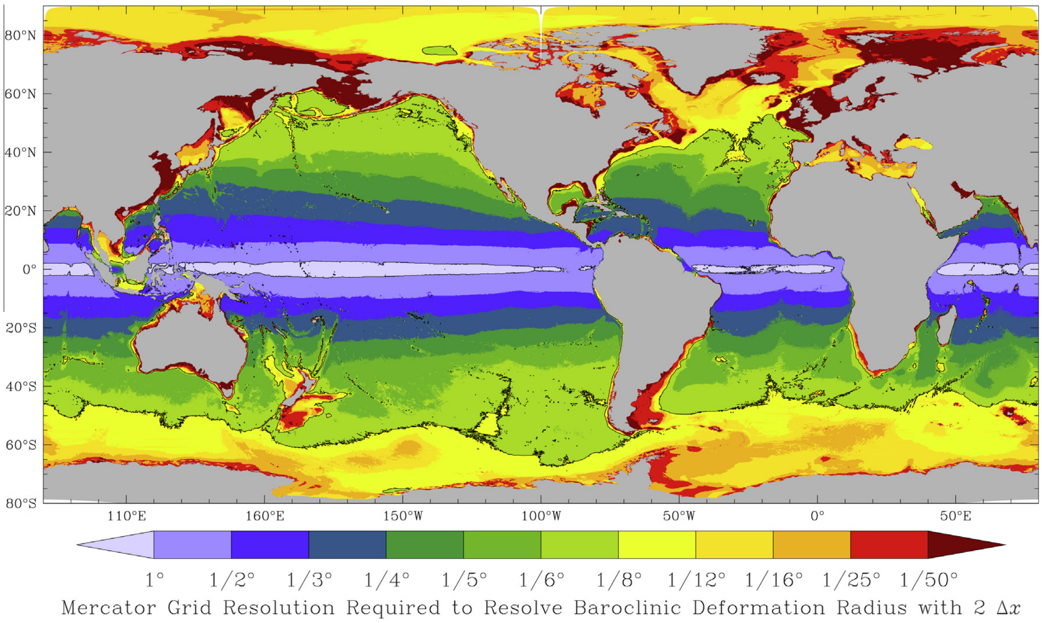


Figure 4.11: The oceanic resolution needed to resolve the Rossby Radius of deformation in an ocean model [from Hallberg et al., 2013].

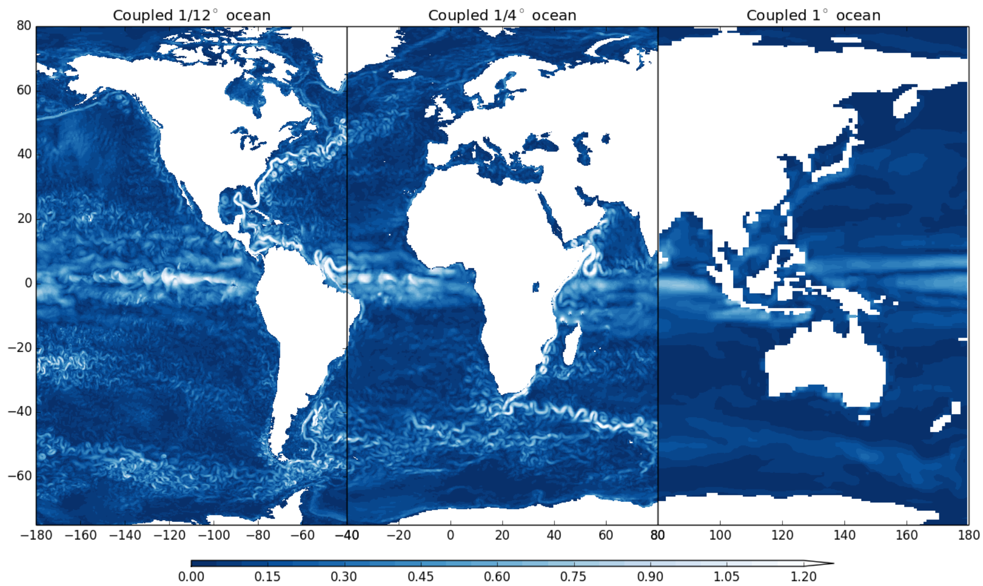


Figure 4.12: The same ocean model at different horizontal resolutions, increasing from right to left.

Let's now go a little ahead of ourselves. Consider that the Coriolis parameter is not constant and is actually a function of latitude $f(y)$. The nondivergent condition $\nabla \cdot (f\mathbf{u}) = 0$ is satisfied by the geostrophically balanced flow. Cross-differentiating the geostrophic equations (4.68) gives

$$\frac{\partial f}{\partial y} v_g + f \nabla_z \cdot \mathbf{u_g} = 0 \tag{4.82}$$

Using mass continuity leads to

$$\boxed{\beta v_g = f \frac{\partial w}{\partial z}}, \tag{4.83}$$

where $\beta \equiv \frac{\partial f}{\partial y}$. This is a geostrophic vorticity balance, also called Sverdrup balance. In a Sverdrup balance, the vertical velocity results from an external agent, most notably wind stress. It states that the vertical shear in the vertical velocity balances a meridional current, with the Coriolis parameter f and the planetary vorticity gradient β determining the sense and strength of the meridional flow. A vertical velocity shear arises when there is a nonzero curl in the wind stress acting on the ocean surface. Vorticity is then transferred to the ocean via **frictional** effects causing *Ekman pumping or suction*. These effects alter the vertical structure of the vertical velocity and, through Sverdrup balance, induce a meridional flow.

4.9 Shallow-water equations

To describe large-scale oceanic, and atmospheric, motions, where the horizontal scale is much larger than the vertical scale, we can use a set of simplified equations that retain the necessary ingredients of the fluid motion but use some useful approximations. We will thus consider a fluid in hydrostatic balance of constant density and, for simplicity, we will also consider a flat bottom. The necessary condition of the shallow-water equations is that the horizontal length scale must be much larger than the vertical scale over which the fluid develops so that $L >> H$.

If the fluid is in hydrostatic balance

$$\frac{\partial p}{\partial z} = -\rho g. \tag{4.84}$$

Then the total pressure will be (assuming constant density)

$$p(x, y, z, t) = -\rho g z + p_0. \tag{4.85}$$

Pressure must vanish at the surface, so that $p = 0$ at $z = \eta$, so that our total pressure will be

$$p(x, y, z) = \rho g(\eta(x, y) - z) \tag{4.86}$$

This means that the horizontal gradient of pressure, and horizontal velocities, is independent of depth

$$\nabla p = \rho g \nabla \eta \tag{4.87}$$

and the horizontal momentum equations reduce to

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p = -g\nabla\eta \tag{4.88}$$

We can now easily add rotation to our shallow-water momentum equations

$$\boxed{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p = -g\nabla\eta} \tag{4.89}$$

Where

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y}. \tag{4.90}$$

and $\boldsymbol{u} = (u, v)$.

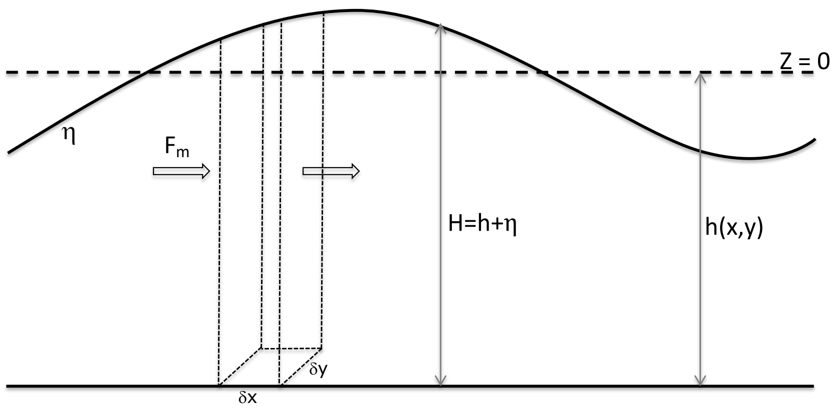


Figure 4.13: Schematic of a flat-bottomed shallow-water system and mass balance within a column of fluid.

The continuity equation is obtained by the mass balance within an infinitesimal column of fluid. The mass flux passing through a section of the column is $F_m = \rho u(h+\eta)\delta y$ and the difference between the fluxes into and out of the section is given by

$$\delta x \delta y \frac{\partial}{\partial x}[\rho u(h+\eta)] \tag{4.91}$$

Considering the total volume, the net rate of change is

$$\frac{\partial}{\partial t}[\rho(h+\eta)] + \frac{\partial}{\partial x}[\rho u(h+\eta)] + \frac{\partial}{\partial y}[\rho v(h+\eta)] = 0 \tag{4.92}$$

Because ρ is constant, the new continuity equation for the shallow-water system is

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0 \tag{4.93}$$

$$\boxed{\frac{\partial \eta}{\partial t} + \nabla \cdot (\boldsymbol{u}H) = 0} \tag{4.94}$$

and if the perturbation is small and h is constant, mass continuity reduces to the linear equation

$$\boxed{\frac{\partial \eta}{\partial t} + \mathrm{h}\nabla \cdot \boldsymbol{u} = 0} \tag{4.95}$$

If there is flux by advection this is balanced by a net increase in mass and an increase in height, giving rise to a vertical velocity, so that the mass convergence is balanced by the increase in height allowing for a dynamical surface elevation. This will be the basis for the propagation of waves within the rotating shallow-water system.

Exercices

1. Use $\phi = p/\rho_0$ and the definition of buoyancy $b = -g\rho/\rho_0$ to rewrite the hydrostatic balance and thermal wind equations.
2. Where is thermal wind velocity directed in the southern hemisphere, considering a poleward increasing (decreasing) density (temperature)? (see Fig. 4.14)
3. How is thermal wind shear changed as we approach the poles?

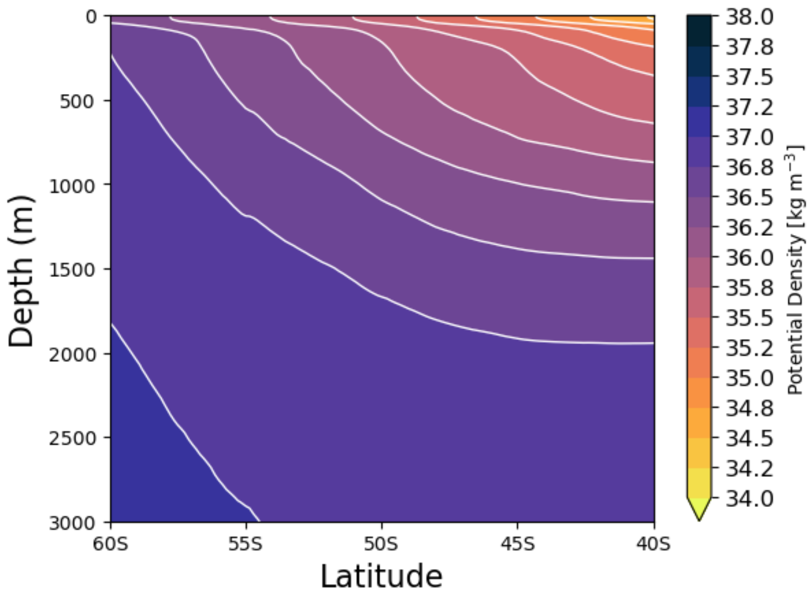


Figure 4.14: Zonal-mean potential density in the latitudes of the Drake Passage.

Bibliography

Kalnay, E., M. Kanamitsu, R. Kistler, W. Collins, D. Deaven, L. Gandin, M. Iredell, S. Saha, G. White, J. Woollen, Y. Zhu, M. Chelliah, W. Ebisuzaki, W. Higgins, J. Janowiak, K. Mo, C. Ropelewski, J. Wang, A. Leetmaa, R. Reynolds, R. Jenne, and D. Joseph, The NCEP/NCAR 40-year reanalysis project, *Bull. Amer. Meteor. Soc.*, *77*, 437–471, 1996.

Kuhlbrodt, T., A. Griesel, M. Montoya, A. Levermann, A. Hofmann, and S. Rahmstorf, On the driving processes of the atlantic meridional overturning circulation, *Reviews of Geophysics*, *45*, RG2001.10.1029/2004RG000,166, 2007.

Olbers, D., J. Willebrand, and C. Eden, *Ocean Dynamics*, Springer-Verlag, 704 pp, 2012.

Talley, L., G. L. Pickard, W. J. Emery, and J. H. Swift, *Descriptive Physical Oceanography*, Academic Press, 560 pp, 2011.

Vallis, G. K., *Atmospheric and Oceanic Fluid Dynamics*, Cambridge Univ. Press, 745 pp, 2006.

Vallis, G. K., *Essentials of Atmospheric and Oceanic Dynamics*, Cambridge Univ. Press, 356 pp, 2019.