PLAYING WITH NUMBERS

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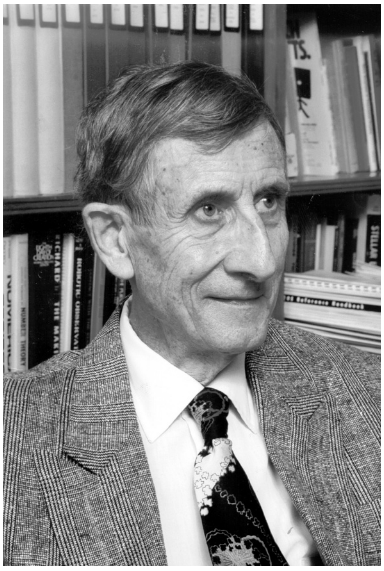

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I write this piece in honor of my friend and hero Abdus Salam, founder and moving spirit of the International Centre for Theoretical Physics. Salam was great as a scientist, greater as an organizer, greatest as the voice of conscience speaking for the advancement of science among the poorer two thirds of mankind. He did not play with numbers. He had more important things to do.

When I was a child, I loved playing with numbers. My interest in science was not driven by any noble desire to understand the mysteries of nature, discover new particles or cure diseases. I never thought deep thoughts about the universe. Science was exciting because it was full of numbers that I could calculate.

When I was fourteen, I won a prize in high-school. The prize was a book that I could choose myself. I chose "The Theory of Numbers", by G.H. Hardy and E.M. Wright, which was then newly published. It is a wonderful book, written at the right level for a bright teen-ager in love with numbers. I read it through from beginning to end. My favorite chapter was chapter 19, with the title "Partitions". A partition of a number n means a set of positive numbers which add up to n. For example, there are five partitions of 4, namely 4, 3+1, 2+2, 2+1+1 and 1+1+1+1. Chapter 19 was full of beautiful theorems about the partition-function p(n), which counts the total number of partitions of n. The function begins with p(1)=1, p(2)=2, p(3)=3, p(4)=5, p(5)=7, p(6)=11. The most beautiful theorems had been discovered by the Indian mathematical genius Ramanujan twenty years earlier. Ramanujan's theorems say that p(5k+4) is divisible by 5, p(7k+5) is divisible by 7, and p(11k+6) is divisible by 11, for any value of k. Ramanujan discovered them by looking at a table of the numbers p(n), and proved them by some clever arguments that are explained in chapter 19. Ramanujan died in 1920 at the age of 32. I recognized him as a kindred spirit, even more in love with numbers than I was.

After reading chapter 19, I took a closer look at the partitions of 4, 9 and 14, the first three numbers of the form 5k+4. I thought, it is all very well for Ramanujan to say that in each case the number of partitions is divisible by 5, but it would be even better if we had a way to divide the partitions into 5 equal sets. That would explain why his theorem is true. I asked the question, whether we can find a number R(P) for each partition P, with the property that the partitions of 5k+4 divide into 5 equal sets C(0), C(1), C(2), C(3), C(4), where C(r) is the set of partitions whose R(P) are of the form 5k+r for r=0,1,2,3,4. I called R(P) the rank of the partition P. I also looked at the partitions of 5 and 12 and tried to find a way of dividing them into 7 equal classes.

Four years later, when I was an undergraduate at Cambridge University, I was still playing around with partitions. After trying many candidates for the rank R(P), I found one that worked. The winning candidate turned out to be simple. The rank is the largest part in P minus the number of parts. Thus the partitions 4, 3+1, 2+2, 2+1+1, 1+1+1+1 of 5 have ranks 3, 1, 0, -1, -3, and there is one in each of the sets C(3), C(1), C(0), C(4), C(2). I checked that this rank also works for the partitions of 9 with 6 partitions in each set, and for the partitions of 14 with 27 partitions in each set. This could not have happened by accident. If you assigned the partitions of 14 by random chance into five sets, the odds against an equal division are greater than ten thousand to one. After I checked the partitions of 9 and 14, I knew that the equal division into five sets by rank must work for the partitions of 5k+4 for all k. Next I looked at the partitions of 5, which are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1, with ranks 4, 2, 1, 0, -1, -2, -4, one in each of the seven sets C(r) with ranks of the form 7k+r. I checked that the rank also works for the partitions of 12, dividing them into seven equal sets with 11 partitions in each set. So I was sure that the rank would work for the partitions of 7k+5 for all k. But the rank did not work for dividing partitions of 11k+6 into eleven equal sets. It failed already for the partitions of 6, since the partitions 3+3 and 4+1+1 of 6 both have rank 1, and there are no partitions with rank 4.

It was quite a triumph for a second-year student to discover something beautiful that Ramanujan had missed. I would have loved to tell Ramanujan about it, but Ramanujan was dead. Two other things made me sad. I could not find an explanation of Ramanujan's theorem for 11k+6, and in spite of long-continued efforts I could not find a proof of the equal division for partitions of 5k+4 and 7k+5. Finally I gave up trying and published my unfinished work in the student magazine "Eureka". My paper, with the title, "Some Guesses in the Theory of Partitions", contained only conjectures and no proofs. This was my first published paper. I stated two conjectures which I called the Rank Conjecture and the Crank Conjecture. The Rank Conjecture said that the partitions of 5k+4 and 7k+5 could be divided by their ranks into five or seven equal sets. The Crank Conjecture said that there must exist some other property of a partition, called the crank, which would divide the partitions of 11k+6 into eleven equal sets. I summarized the numerical evidence that supported both conjectures. But I had no idea how to find a plausible candidate for the crank.

This story has a happy ending. Eleven years after my paper was published, Oliver Atkin and Peter Swinnerton-Dyer, two mathematician friends of mine, proved the Rank Conjecture, using some powerful ideas invented by Ramanujan. Thirty-four years after that, the Crank Conjecture was proved by two other friends, George Andrews and Frank Garvan. Andrews and Garvan proved that the same crank will divide the partitions equally for all three cases, 5k+4, 7k+5 and 11k+6. The definition that they found for the crank is weird. Suppose that a partition has s parts. Let t be the biggest part minus the second biggest, and let d be the t+1'th biggest part minus t, with the convention that the t+1'th biggest part is zero if t+1>s. Then the crank is defined to be d if t>0 and s if t=0. It is easy to check that this crank works for the partitions of 6. The partitions are 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1, with cranks -6, -4, -2, -3, 2, 1, -1, 3,4,0,6, one belonging to each of the eleven sets C(r). Andrews and Garvan found this definition by following a long trail of analysis also originating from Ramanujan. I would never have found it using my pedestrian method of random search. I was lucky to live long enough to see my conjecture proved after forty-five years.

All my life, I have worked as a scientist looking for situations where a little elegant mathematics can help us to understand nature. I found problems that I could solve with a teaspoonful of elegant mathematics, in physics and engineering and astronomy and biology. I never worried whether the problems were important or unimportant. So long as the mathematics was beautiful, I was happy. My work on the Ramanujan partition theorems was the least important of all, and the most beautiful. That was where my life as a scientist started, with a school prize at age fourteen. Playing with numbers was a good way to start.

References

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