SUPERCONFORMAL FIELD THEORIES

John H. Schwarz

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Introduction

One reason that superconformal field theories are particularly interesting is their role in AdS/CFT duality. There are three maximally supersymmetric examples:

- M theory on $AdS_4 \times S^7$ is dual to a SCFT in 3d.
- Type IIB superstring theory on $AdS_5 \times S^5$ is dual to a SCFT in 4d ($\mathcal{N} = 4$ SYM theory).
- M theory on $AdS_7 \times S^4$ is dual to a SCFT in 6d.
Conformal Symmetry

In two dimensions the conformal group is infinite dimensional, whereas for $d > 2$ it is $SO(d, 2)$. This talk will only consider $d > 2$.

In supersymmetric theories the conformal group is extended to the superconformal group, which is a supergroup. For three dimensions it is $OSp(N|4)$; in four dimensions it is $SU(2, 2|\mathcal{N})$ for $\mathcal{N} < 4$ and $PSU(2, 2|4)$ for $\mathcal{N} = 4$; in six dimensions it is $OSp(6, 2|\mathcal{N})$. 
The Type IIB / $\mathcal{N} = 4$ SYM Example

Type IIB superstring theory and $SU(N)\mathcal{N} = 4$ SYM theory are both very well understood, so the $AdS_5 \times S^5$ duality has been extensively studied. The 't Hooft parameter of the gauge theory, $\lambda = g_{YM}^2 N$, corresponds to $R^4$ measured in string units, where $R$ is the radius of the sphere (and the AdS). There are also $N$ units of five-form flux:

$$\int_{S^5} F_5 \sim N$$
The Two M-Theory Dualities

These are much tougher: M-theory is less well understood than type IIB superstring theory. Also, the dual SCFTs in 3d and 6d are much more elusive.

For a long time it was believed that these 3d and 6d theories do not have a Lagrangian description. Then a Lagrangian description of the 3d theory was constructed (the ABJM theory). I plan to discuss how the ABJM construction overcame the apparent obstructions. Then I will discuss the status of the 6d theory.
When does a QFT have a Lagrangian?

We are accustomed to defining quantum theories by “quantizing” classical theories, but this only makes sense for theories that have a classical limit. This is the case whenever it is possible to define a weak coupling limit. However, some theories are always strongly coupled.

If the interaction strength varies with energy scale, then there is a weak coupling limit in either the IR or the UV, which has a Lagrangian description. So a non-Lagrangian QFT must be a CFT (plus possible soft terms such as masses).
Some CFTs (such as $\mathcal{N} = 4$ SYM) have moduli that can be continuously varied to define a weak coupling limit. Such theories also have Lagrangian descriptions. This includes any 4d Yang-Mills CFT.

There are SCFTs in 4d that are not Yang–Mills theories and are always strongly coupled. Examples include a class of $\mathcal{N} = 2$ theories called $T_N$, with $N = 3, 4, \ldots$ It is believed that they do not have Lagrangian descriptions, though this has not been proved.
SCFTs in Three Dimensions

String theories contain a dilaton field whose expectation value gives the string coupling constant, which corresponds to $g_{YM}$ in the dual CFT. By contrast M-theory has no dilaton. Hence the dual SCFTs in 3d and 6d have no variable coupling constants.

Despite this, I proposed in 2004 that the 3d SCFT that is dual to M-theory on $AdS_4 \times S^7$ could be a gauge theory in which the YM fields appear in Chern–Simons terms. However, I did not find the correct theory.
In addition to the non-dynamical YM gauge fields, this 3d $\mathcal{N} = 8$ SCFT should contain dynamical scalars $\Phi$ and spinors $\Psi$ transforming as inequivalent $8$-s of the $Spin(8)$ R-symmetry subgroup of the superconformal supergroup $OSp(8|4)$.

The dimension of $\Phi$ is $1/2$ and the dimension of $\Psi$ is $1$, so the possible dimension-$3$ terms are schematically of the form:

$$(D\Phi)^2, \quad \Psi D\Psi, \quad \Phi^2 \Psi^2, \quad \Phi^6.$$
A theory with these properties and $SO(4)$ gauge symmetry was constructed by Bagger and Lambert and independently by Gustavsson in 2007. Ironically, they discovered a beautiful theory that is not the desired dual of M-theory on $AdS_4 \times S^7$. Its role, if any, in string theory/M-theory is still unclear.

The definitive paper by Aharony, Bergman, Jafferis, and Maldacena (ABJM), which gives the desired dual SCFT, appeared in June 2008.
Orbifolding the Sphere

A key step to understanding why a Lagrangian theory exists is to understand the dual interpretation of the Chern–Simons level \( k \). It corresponds to orbifolding the \( S^7 \) by a cyclic group \( \mathbb{Z}_k \).

This is most easily understood by embedding the \( S^7 \) in \( \mathbb{C}^4 \) and forming the \( \mathbb{C}^4/\mathbb{Z}_k \) orbifold by the identifications

\[ z^a \sim \exp(2\pi i/k)z^a, \quad a = 1, 2, 3, 4 \]
The orbifold action breaks the $SO(8)$ R-symmetry to an $SU(4) \sim SO(6)$ subgroup when $k > 2$. For $k = 1, 2$ the $SO(8)$ R-symmetry is unbroken.

Accordingly, the full isometry group of the $AdS_4 \times S^7/\mathbb{Z}_k$ solution of M-theory is

$$OSP(8|4) \quad \text{for} \quad k = 1, 2$$

$$OSP(6|4) \quad \text{for} \quad k > 2$$

Thus there is only $\mathcal{N} = 6$ supersymmetry for $k > 2$. The same is required of the dual SCFTs.
The ABJM Theories

ABJM constructed $\mathcal{N} = 6$ superconformal Chern–Simons theories with the gauge group $U(N) \times U(N)$ and bifundamental matter. $N$ corresponds to the number of flux units $\int *F_4$ on $S^7/\mathbb{Z}_k$. The radius $R$ of the AdS and the sphere is given by $kN \sim R^6$.

The ABJM construction of the $\mathcal{N} = 6$ SCFT also works for $SU(N) \times SU(N)$, but the extra $U(1)$ factors are required to properly implement the orbifold action in the gauge theory. In fact, the $SU(2) \times SU(2)$ case is the BLG theory.
The original problem we set out to solve has \( k = 1 \), but it is now viewed as one member of a family of theories, labeled by \( k \), which are weakly coupled for large \( k \). This is why the Lagrangian exists. Moreover, the \( \mathcal{N} = 6 \) symmetry of the classical theory is enhanced to \( \mathcal{N} = 8 \) for \( k = 1, 2 \) by quantum effects!

The ABJM SCFT has a perturbation expansion in powers of \( 1/k \). Also, for fixed ’t Hooft parameter, \( \lambda = N/k \), it has a large-\( N \) expansion in powers of \( 1/N \). This has the usual topological interpretation: The leading term is the planar approximation, etc.
Lessons for 6d Superconformal Theories

This is a much tougher case: For one thing, the 6d theory should contain 2-form fields with self-dual field strengths, which are awkward to work with. One way to evade this issue is to focus on equations of motion. A bigger problem is how to make such fields non-abelian.

There was an interesting attempt by Lambert et al. recently, which constructed some intriguing equations of motion. However, as the authors point out, their 3-form field strengths cannot be expressed in terms of two-form potentials. So this was unsuccessful.
We need an analog of the supersymmetric orbifolding, which was so crucial in the $AdS_4 \times S^7$ case. This made it possible to define a weak coupling limit, which ensured the existence of a classical description of the theory. Orbifolding the sphere in $AdS_7 \times S^4$ does not work.

Is it possible to orbifold the $AdS_7$ rather than the sphere? This would introduce an integer $k$, the analog of the Chern–Simons level in the the ABJM theory, as a parameter of the 6d geometry. It would also break the conformal symmetry. This approach is being explored.
In recent years, Gaiotto and others have taught us a great deal about the 4d $\mathcal{N} = 2$ SCFTs (plus mass terms) that arise from wrapping the unknown 6d theory on a Riemann surface with punctures.

If we had a good description of the 6d theory, it would be a very powerful tool for understanding these 4d theories better. It might even be helpful for understanding M-theory better.
Conclusions

• Some SCFTs that have no weak coupling limit might not have a Lagrangian description.

• The 3d SCFT that is dual to M-theory on $AdS_4 \times S^7$ has a Lagrangian description (the ABJM theory).

• There has been no significant progress in formulating the 6d SCFT dual to M-theory on $AdS_7 \times S^4$. There might not be a classical Lagrangian in this case, but I think we should keep trying.

THE END